INDECOMPOSABLE SOERGEL BIMODULES FOR UNIVERSAL COXETER GROUPS

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ABSTRACT. We produce an explicit recursive formula which computes the idempotent projecting to any indecomposable Soergel bimodule for a universal Coxeter system. This gives the exact set of primes for which the positive characteristic analogue of Soergel's conjecture holds. Along the way, we introduce the multicolored Temperley-Lieb algebra.

CONTENTS

1. INTRODUCTION

Most of this paper is devoted to the introduction and the elementary representation theory of a certain 2-category, the *multi-colored Temperley-Lieb 2-category*, defined in §2. It is a straightforward generalization of the more familiar Temperley-Lieb category appearing in the representation theory of $\mathfrak{sl}_2$, though beyond the most natural case of two colors, it seems not to have appeared previously in the literature. In earlier work of the first author [5], a connection was made between the two-colored Temperley-Lieb 2-category and the category of Soergel bimodules for the infinite dihedral group. The representation theory of the Temperley-Lieb algebra was then used to prove facts about these Soergel bimodules, leading to answers for certain basic questions in positive characteristic Kazhdan-Lusztig theory. In this paper, we make a connection between the multi-colored Temperley-Lieb 2-category and the category of Soergel bimodules for universal Coxeter groups, and prove the analogous results in Kazhdan-Lusztig theory. The remainder of the introduction will fill in the details.

1.1. **Kazhdan-Lusztig theory in positive characteristic.** In the year 1979, Kazhdan and Lusztig (abbreviated "KL") introduced their celebrated *KL polynomials* for any Coxeter system [13]. These polynomials, living as coefficients in the Iwahori-Hecke algebra, have become a fundamental tool in representation theory, geometry, and combinatorics. However, they are also a fundamental mystery. Despite countless papers exploring the combinatorics of KL polynomials, very little is known outside of specific cases. The only infinite families of Coxeter groups for which we have a complete understanding of KL polynomials are the dihedral groups (a simple exercise) and the universal Coxeter groups (a result of Dyer [4]). Recall that universal Coxeter groups are groups generated by involutions, with no other relations.

As we will discuss shortly, KL polynomials encode multiplicities attached to certain important categories (representation-theoretic, geometric, or otherwise) defined in characteristic zero. For crystallographic Coxeter groups, one can choose an integral form of these categories, in order to define their characteristic p analogs. One can encode the new multiplicities in so-called *p-KL polynomials*, which depend strongly on the specific value of p, and eventually (for p large) agree with the ordinary KL polynomials. Far less is known about the p-KL polynomials, nor is there a known algorithm to compute them within the Hecke algebra, as there is for ordinary KL polynomials. The following question is already of great interest.

Question 1.1. Given a crystallographic Coxeter group W and a prime p, for which $w \in W$ does there exist a $y \leq w$ such that the p-KL polynomial $h^p_{y,w} \in \mathbb{Z}[v, v^{-1}]$ disagrees with the ordinary KL polynomial $h_{y,w}$?

For example, recent work of Williamson [27] has found an infinite family of such quadruples (W, p, w, y) in type A, refuting a well-known conjecture about Lusztig's character formula. Answering this question for Weyl groups and affine Weyl groups would have significant import for modular representation theory (see [21]).

In order to make sense of the p-KL polynomial as encoding multiplicities, we must specify in which category we work. The name p-KL polynomial originally referred to multiplicities in the category of *parity sheaves* [12]. Ultimately, we will use the diagrammatic category $\mathcal{D}$ defined by the first author and Williamson in [8], though there are connections between $\mathcal{D}$, parity sheaves, and the category of Soergel bimodules $\mathcal{B}$ introduced by Soergel [22]. We shall motivate these connections in the introduction, and work in §3 entirely with $\mathcal{D}$.

Question 1.1 then becomes a question about how "big" the indecomposable objects in $\mathcal{D}$ are. For the baby case of universal Coxeter groups, we construct the indecomposable objects explicitly in the generic case, as the images of certain idempotents (already a new result in characteristic zero). By identifying the denominators in these idempotents, we determine which finite characteristics will deviate from the generic behavior, thus answering Question 1.1. Similar results for the other baby case, dihedral groups, can be easily extrapolated from the first author's work [5]. This is a small step along a very long and difficult road.

1.2. **Soergel bimodules and Soergel diagrammatics.** In 1992, Soergel [20] introduced an additive category $\mathcal{B} = \mathcal{B}(W, S, V, \Bbbk)$ of graded bimodules over a polynomial ring, whose objects have come to be known as *Soergel bimodules*. This category depends on a Coxeter system (W, S), a field $\Bbbk$, and a finite dimensional representation V of W over $\Bbbk$ (see [22] for this general definition). One important special case will be when $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and $V = V_{\text{rootic}}$ is the *rootic representation*[1] of (W, S). When W is crystallographic, its rootic representation can be defined over $\mathbb{Z}$, and thus over any field.

The motivation for introducing $\mathcal{B}$ is that, when W is a Weyl group and V its rootic representation in characteristic zero, there is an isomorphism between (a simplified version of) $\mathcal{B}$ and additive subcategories of the representation-theoretic and geometric categories which KL polynomials study. More precisely, this simplified version is equivalent to the projective objects in the principal block of the BGG category $\mathcal{O}$, or the semisimple N-equivariant perverse sheaves on the flag variety G/B. Here G is a connected reductive complex algebraic group, B denotes a Borel subgroup of G, and N denotes the unipotent radical of B. One advantage of Soergel's approach is that $\mathcal{B}$ can be defined for any Coxeter group, even non-crystallographic groups for which there is no corresponding geometry or representation theory. Another advantage is that $\mathcal{B}$ has a simple algebraic definition, allowing one to study KL theory using low-tech methods.

For our purposes, the relevant feature of $\mathcal{B}$ (and, in a sense, of the other categories as well) is *Soergel's categorification theorem* ([20],[22]), which states that Soergel bimodules are a categorification of the Hecke algebra $\mathbf{H}(W)$ of W. In other words, there is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

$$\text{ch} : [\mathcal{B}(W, S, V, \Bbbk)] \longrightarrow \mathbf{H}(W) \tag{1.1}$$

from the split Grothendieck group to the Hecke algebra. Soergel proved this result for any Coxeter group, and for any representation V which is *reflection vector faithful* (see [22] for the definition) over a infinite field $\Bbbk$ of characteristic $\neq 2$.[2] The indecomposable objects $\{B_x\}_{x \in W}$ in $\mathcal{B}$ are classified by elements of W, and they descend to some *positive* basis $\{\text{ch}[B_x]\}$ of the Hecke algebra (i.e. certain coefficients are positive). The Hecke algebra possesses a natural (positive) basis, the KL basis $\{b_x\}_{w \in W}$ (encoded by the KL polynomials). This raises the following question.

Question 1.2. Given $(W, S, V, \Bbbk)$ with V reflection vector faithful, for which $w \in W$ will it be the case that $\text{ch}[B_w] = b_w$?

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[1]The rootic representation is called the *geometric representation* in Bourbaki [2] or Humphreys [11]. The notational preference is explained in [17].

[2]Unfortunately, V_{rootic} is not always reflection vector faithful. However, when $\Bbbk = \mathbb{R}$, Soergel constructed an explicit reflection vector faithful representation analogous to V_{rootic}, and the second author [18] has given numerous theorems relating results for V_{rootic} to results for this explicit representation.

When V is reflection vector faithful and $\Bbbk$ has characteristic 0, it was conjectured by Soergel that every $w \in W$ has this property. When W is a Weyl group, this conjecture is equivalent to the famed Kazhdan-Lusztig conjecture, proven by Brylinski-Kashiwara [3] and Beilinson-Bernstein [1] using difficult geometric techniques. Soergel hoped that the algebraic setting of $\mathcal{B}$ would allow for a simpler solution. The baby case of the dihedral group was proven by Soergel [20]. The universal Coxeter group case was done by Fiebig in [9], using the tool of moment graphs. It was also shown later by the second author in unpublished work, by constructing idempotents using singular Soergel bimodules [28]. The general case was recently proven by the first author and Williamson [7] for V_{rootic} when $\Bbbk = \mathbb{R}$; therefore, the KL basis b_w really does encode something about characteristic 0 Soergel bimodules.

Soergel's categorification theorem implies that $\mathcal{B}$ is the correct object to study in finite odd characteristic, so long as V is reflection vector faithful. When W is a Weyl group, V_{rootic} will be reflection vector faithful in characteristic $\neq 2$, and Question 1.1 is equivalent to Question 1.2. However, an infinite Coxeter group does not possess a faithful representation in positive characteristic, so that $\mathcal{B}$ is not quite the correct category to study.

It is somewhat naive to assume that using the same definitions in characteristic p will yield a category with similar properties. The most appropriate way to define a finite characteristic analog of an additive (resp. abelian) $\mathbb{R}$-linear category is to first choose an integral form. This involves finding a projective generator P and a $\mathbb{Z}$-algebra E such that $E \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{End}(P)$. Then one considers the category of projective (resp. all) $E \otimes_{\mathbb{Z}} \Bbbk$-modules for other fields $\Bbbk$. A typical choice for a generator would be the sum $P_{\text{min}} = \oplus_{w \in W} B_w$ of all the indecomposable objects, but this choice makes computing E quite difficult. In fact, the crux of Soergel's construction is that $\mathcal{B}$ (and category $\mathcal{O}$ and perverse sheaves) has a nice combinatorial generator P_{BS}, the sum of all the *Bott-Samelson objects*. Soergel bimodules are, by definition, summands of Bott-Samelson bimodules, which in turn admit a simple description. In [8], the first author and Williamson show that the endomorphism algebra $\mathrm{End}(P_{\text{BS}})$ also admits a nice combinatorial description, using so-called *Soergel diagrams*.

In [8] one defines a diagrammatic category $\mathcal{D}$ depending on a *realization*, which is roughly the data of $(W, S, V, \Bbbk)$ together with a choice of simple roots and coroots (although $\Bbbk$ can be any commutative ring). There is an equivalence $\mathcal{D} \cong \mathcal{B}$ of monoidal categories when the latter is "well behaved", i.e. when (1.1) gives an isomorphism and the indecomposable objects are parametrized by W. Under some minimal assumptions, $\mathcal{D}$ is well behaved in this sense even when $\mathcal{B}$ is not (such as when the representation is not reflection vector faithful), justifying the statement that $\mathcal{D}$ is the appropriate replacement for $\mathcal{B}$. We still denote the indecomposable objects of $\mathcal{D}$ by B_w for $w \in W$.

Question 1.3. Given a realization over a complete local ring $\Bbbk$ where $[\mathcal{D}] \cong \mathbf{H}(W)$, for which $w \in W$ will it be the case that $\mathrm{ch}[B_w] = b_w$?

This is the most general alternative to Question 1.1, and it depends on the realization itself, not just on the characteristic of $\Bbbk$. When W is crystallographic and the realization is rootic in finite characteristic, $\mathcal{D}$ agrees with the category of parity sheaves [12] on the flag variety. Parity sheaves are a finite characteristic analog of perverse sheaves[3], whose multiplicities were originally called p-KL polynomials. Therefore, Question 1.1 is a special case of Question 1.3.

Having chosen P_{BS} rather than $P_{\min}$ as our generator, one has an implicit definition of the indecomposable objects as certain summands of Bott-Samelson objects. Finding an explicit construction of each B_w is the true goal underlying all these Grothendieck-group-theoretic questions about their sizes. There is a general computational algorithm for the idempotents (inside an endomorphism ring of a Bott-Samelson object) which project to each indecomposable summand (see [17]), but this algorithm is unsatisfactory in that it provides no insight into the dependence of this idempotent on the choice of realization. Instead, one hopes for an explicit formula (possibly inductive) for the idempotents in the generic case, yielding explicit knowledge of their denominators. This is what we achieve for universal Coxeter groups. The case of a general Coxeter group seems to be drastically more difficult.

Example 1.4. One can consider the most standard realization of a universal Coxeter group, arising from a symmetric Cartan matrix where each diagonal entry is 2 and each non-diagonal entry is -2. This realization can be defined over $\mathbb{Z}$, and thus over any field $\Bbbk$, though only the characteristic of $\Bbbk$ is relevant here. Each element of the Coxeter group has a unique reduced expression, having the form $w = s_1 s_2 s_3 \cdots s_d$ where s_i are simple reflections and $s_i \neq s_{i+1}$. A maximal alternating subsequence of this reduced expression is a consecutive subsequence $s_i s_{i+1} \cdots s_{i+k}$ (having length $k+1$), satisfying $s_i = s_{i+2} = \cdots$ and $s_{i+1} = s_{i+3} = \cdots$, and which is not contained in a larger alternating consecutive subsequence. Our results (c.f. Proposition 2.22) state that $\mathrm{ch}[B_w] = b_w$ if and only if the binomial coefficients $\binom{k}{m}$ are invertible in $\Bbbk$ for each $0 \leq m \leq k$, for every k such that $k+1$ is the length of a maximal alternating subsequence of w.

1.3. **Techniques.** In [5], the first author demonstrated that Soergel bimodules for the infinite dihedral group were intimately related to the Temperley-Lieb algebra which arises in $\mathfrak{sl}_2$ representation theory. The familiar Jones-Wenzl idempotents in the Temperley-Lieb algebra were transformed into idempotent endomorphisms of Bott-Samelson bimodules, projecting to the indecomposable summands. This paper takes these ideas to their natural conclusion, producing a relationship between the multicolored Temperley-Lieb 2-category and Bott-Samelson bimodules (or rather, their diagrammatic analogues) for the corresponding universal Coxeter group.

[3]Perverse sheaves do exist in finite characteristic, but like Soergel bimodules for non-reflection-faithful representations, they do not possess the desired categorification-related properties. Parity sheaves do.

In §2 we define the multicolored Temperley-Lieb 2-category and explore its representation theory. We define the analogues of Jones-Wenzl idempotents. We provide a recursive formula for these idempotents, allowing one to categorify Dyer's inductive formula for the KL basis. We also provide an immediate formula for these idempotents in terms of the Jones-Wenzl idempotents in the usual Temperley-Lieb algebra (which unfortunately have no easy closed formula, though see [19]). This second formula implies a criterion for when the idempotent is not defined, which will lead to the answer to Question 1.3. Specifically, $\mathrm{ch}[B_w] = b_w$ so long as certain "colored quantum binomial coefficients" are invertible.

This answer to Question 1.3 relies on the fact that Jones-Wenzl projectors in the usual Temperley-Lieb algebra exist if and only if certain quantum binomial coefficients are invertible. Though fundamental to the theory of Temperley-Lieb algebras, this fact does not seem to appear in the literature. To remedy this, we have included an appendix written by Ben Webster, with a proof of the result over a general ring (see Theorem A.2).

In §3 we define the diagrammatic category $\mathcal{D}$ associated to the most general realization of a universal Coxeter group. Our definition is purely diagrammatic, using the results of [8], and thus we never mention Soergel bimodules. We prove the main theorem: that the multicolored Temperley-Lieb 2-category encodes all the morphisms of minimal degree in $\mathcal{D}$. Therefore, the Jones-Wenzl analogues provide all the indecomposable idempotents in $\mathcal{D}$.

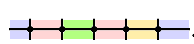
2. The n-colored Temperley-Lieb 2-category

2.1. **Definitions.**

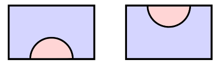
We assume that the reader is familiar with several topics, for which we give some references. Introductory material on the Temperley-Lieb category can be found in [25, 10]. An introduction to (strict) 2-categories and their diagrammatic presentations can be found in [15, section 2]. An introduction to Karoubi envelopes (of categories and 2-categories) can be found in [14].

Let S be a finite set with size n. We associate a color to each element of S, blue to b and red to r, etcetera. Let δ be an indeterminate.

Definition 2.1. The *S-colored* or *n-colored Temperley-Lieb 2-category* $S\mathcal{TL}$ is the $\mathbb{Z}[\delta]$-linear 2-category with objects S, having the following presentation. There is a generating 1-morphism from b to r, for each pair of distinct elements $b \neq r \in S$. Therefore, a general 1-morphism can be represented uniquely by the (non-empty) sequence $\underline{x} = s_1 s_2 \ldots s_m$ of colors through which it passes, satisfying $s_i \neq s_{i+1}$ for all i. We read 1-morphisms from right to left, so that $\underline{x}$ has source s_m and target s_1. We say the 1-morphism has *length* $\ell(\underline{x}) = m$ (this is not additive under composition; it would be additive if we set $\ell(\underline{x}) = m-1$ instead). For instance, the identity 1-morphism of any object $s \in S$ has length 1. We represent a composition of 1-morphisms diagrammatically as a sequence of dots on the line, separating regions of different colors.

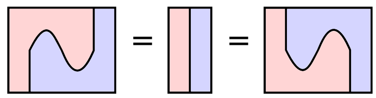
Example 2.2. The 1-morphism $brgryb$: .

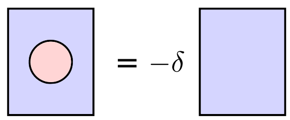
The 2-morphisms are generated by colored cups and caps. More precisely, for each $b \in S$ and for each $r \in S \setminus b$ there is a cap map $brb \to b$ and a cup map $b \to brb$, as pictured below.



In these diagrams, morphisms are read from bottom to top.

There are two types of relations, which hold for every possible coloring of regions.

(2.1) 

(2.2) 

This ends the definition.

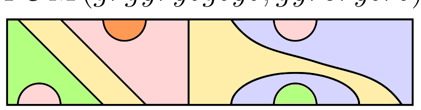
Remark 2.3. We shall actually be interested in a generalization of this definition, introduced in section 2.6. For pedagogical reasons, however, we shall temporarily work with this more familiar-looking definition.

Let $CM(m,k)$ denote the set of (m,k)-crossingless matchings in the planar strip (see [25, section 1] for the definition). Given any element of $CM(m,k)$, one can color the regions by elements of S so that no two adjacent regions have the same color. The resulting diagram will represent some 2-morphism in $S\mathcal{TL}$. Conversely, every 2-morphism in $S\mathcal{TL}$ is a $\mathbb{Z}[\delta]$-linear combination of such colored crossingless matchings.

Definition 2.4. For fixed 1-morphisms $\underline{x} = s_1 s_2 \dots s_{m+1}$ and $\underline{y} = t_1 t_2 \dots t_{k+1}$, we let $CM(\underline{x}, \underline{y})$ denote the subset of (m,k)-crossingless matchings which can be consistently colored to yield a 2-morphism in $\mathrm{Hom}(\underline{x}, \underline{y})$.

For example, $CM(\underline{x}, \underline{y}) = \emptyset$ unless $s_1 = t_1$ and $s_{m+1} = t_{k+1}$.

Example 2.5. An element of $CM(grgyrybgbyb, gyrorybrb)$:



The reader can verify that only four elements of $CM(10,8)$ actually give rise to an element of $CM(grgyrybgbyb, gyrorybrb)$.

Lemma 2.6. *The set $CM(\underline{x}, \underline{y})$ forms a $\mathbb{Z}[\delta]$-basis for $\mathrm{Hom}(\underline{x}, \underline{y})$.*

Proof. This can be proven in exactly the same way that one proves that $CM(m,k)$ is a $\mathbb{Z}[\delta]$-basis for $\mathrm{Hom}(m,k)$ in the usual Temperley-Lieb category. Here is a sketch of such a proof.

The generators of $\mathcal{STL}$ allow one to construct a morphism in $\mathrm{Hom}(\underline{x}, \underline{y})$ for each planar 1-manifold with boundary, with appropriate coloring on the boundary. Relation (2.1) implies that any two isotypic 1-manifolds are equal. Relation (2.2) is equivalent to a family of relations which states that every planar 1-manifold is equal to $(-\delta)^k$ times the underlying crossingless matching, where k is the number of loops removed. It is clear that this family of relations can not create any linear dependencies between crossingless matchings. □

Example 2.7. Note that $CM(\underline{x}, \underline{x})$ always contains the identity crossingless matching $\mathbb{1}$, but may contain no others. For instance, if $s_i \neq s_{i+2}$ for all i, then there can be no cups or caps, and therefore $CM(\underline{x}, \underline{x})$ only contains the identity.

Example 2.8. When $n = 1$ and $S = \{b\}$, the 2-category $\mathcal{STL}$ is very boring, having a unique 1-morphism b with $\mathrm{End}(b) = \mathbb{Z}[\delta]$.

Example 2.9. When $n = 2$ and $S = \{r, b\}$, the 1-morphisms are alternating sequences $\underline{x} = rbrb\ldots$. When $\underline{x}$ and $\underline{y}$ both begin with r, and have lengths $m + 1$ and $k + 1$ respectively, then $CM(\underline{x}, \underline{y}) = CM(m, k)$. In fact, there is an equivalence of categories between the usual Temperley-Lieb category and the full subcategory of $\mathcal{STL}$ obtained by considering only 1-morphisms beginning with r. In many senses, the two-colored Temperley-Lieb category is more natural than the usual Temperley-Lieb category, because the representation theory of $\mathfrak{sl}_2$ is naturally $\mathbb{Z}/2\mathbb{Z}$ graded (even and odd representations), where we identify $\mathbb{Z}/2\mathbb{Z}$ with the quotient $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}}$ of the integral weight lattice by the root lattice. For more on this, see [6].

Exercise 2.10. Conversely, let S be arbitrary, and suppose that $\underline{x}$ and $\underline{y}$ begin with r and have lengths $m + 1$ and $k + 1$ repsectively. Then one has an equality $CM(\underline{x}, \underline{y}) = CM(m, k)$ if and only if both spaces are empty (i.e. $k + m$ is odd), or $\underline{x}$ and $\underline{y}$ both alternate between r and another color b.

By flipping diagrams upside-down, one obtains a bijection between $CM(\underline{x}, \underline{y})$ and $CM(\underline{y}, \underline{x})$. This extends to an antiinvolution ι on $\mathcal{STL}$.

2.2. **The Karoubi envelope.** In this paper, $\Bbbk$ will always be a commutative ring, perhaps with extra structure. In this section and the next, $\Bbbk$ will be a complete local $\mathbb{Z}[\delta]$-algebra. We now work in the 2-category $\mathcal{STL} \otimes_{\mathbb{Z}[\delta]} \Bbbk$ obtained by base change, and abusively denote this category $\mathcal{STL}$.

Remark 2.11. It is well-known that the usual Temperley-Lieb category is cellular (see [26, section 2] for the definition). In fact, it is an especially nice kind of cellular category known as an *object-adapted cellular category*, meaning roughly that the cells correspond to some objects in the category, and that the top cell of each of these objects contains only the identity map. The monoidal structure is usually ignored when studying the cellular structure (certainly the theory of monoidal cellular categories has not been thoroughly explored).

Similarly, the 2-category $S\mathcal{TL}$, when viewed as a 1-category by forgetting the structure of horizontal concatenation, is an (object-adapted) cellular category, using a direct adaptation of the structure on the usual Temperley-Lieb category. The features of the Karoubi envelope of $S\mathcal{TL}$ that we discuss below are in fact rather general properties of object-adapted cellular categories, but we give complete proofs. In particular, references to "shorter sequences" below should be replaced with references to the cellular partial order. Future work of the first author will contain more discussion of object-adapted cellular categories.

Fix a 1-morphism $\underline{x}$ of length $m+1$. A key property of the set $CM(m,m)$, which we used implicitly in Example 2.7, is that every diagram except the identity contains a cap on bottom and a cup on top. In particular, the span of the non-identity diagrams in $CM(\underline{x},\underline{x})$ forms a two-sided ideal $I_{<\underline{x}} \subset \mathrm{End}(\underline{x})$, whose quotient is free of rank 1 over $\Bbbk$, spanned by the identity.

Suppose that one can decompose $\mathbb{1} \in \mathrm{End}(\underline{x})$ into a sum $\mathbb{1} = \sum e_i$ of orthogonal indecomposable idempotents. It is easy to see, by working modulo $I_{<\underline{x}}$, that there is a unique idempotent e_0 with a non-zero coefficient of the identity (in the basis $CM(\underline{x},\underline{x})$), and this coefficient is 1. The remaining idempotents lie within $I_{<\underline{x}}$. Our goal is to prove that within the Karoubi envelope $\mathbf{Kar}(S\mathcal{TL})$, the object $\underline{x}$ has a unique indecomposable summand $V_{\underline{x}}$ which is not a summand of $\underline{y}$ for any shorter sequence; it is the image of e_0. In other words, the idempotents within $I_{<\underline{x}}$ actually factor through shorter sequences $\underline{y}$.

Lemma 2.12. *Suppose that $\Bbbk$ is a complete local ring. Then **Kar**$(S\mathcal{TL})$ has the Krull-Schmidt property.*

Proof. This is a general fact for $\Bbbk$-linear categories with finite dimensional Hom spaces. A similar proof can be found in [16, Proposition 1.1]. □

Proposition 2.13. *Suppose that $\Bbbk$ is a complete local ring. For each sequence $\underline{x}$ choose a decomposition $\mathbb{1} = \sum e_i$ into orthogonal indecomposable idempotents, such that $e_0 = \mathbb{1}$ modulo $I_{<\underline{x}}$. Let $V_{\underline{x}}$ denote the image of e_0, an object in **Kar**$(S\mathcal{TL})$. Then the collection of all $V_{\underline{x}}$ over all sequences $\underline{x}$ form a complete list of non-isomorphic indecomposable objects in **Kar**$(S\mathcal{TL})$, and*

$$\underline{x} \cong V_{\underline{x}} \oplus \bigoplus_{\ell(\underline{y})<\ell(\underline{x})} V_{\underline{y}}^{\oplus m_{\underline{y}}}.$$

In particular, by the Krull-Schmidt property, the object $V_{\underline{x}}$ is independent of the choice of idempotent decomposition, up to isomorphism.

The sections which follow will give a more intuitive and obvious proof under some additional assumptions, and the novice reader should skip there. We now provide a general proof, which is adapted directly from the proof of the Soergel Categorification Theorem found in [8, section 6.6].

Proof. It is not hard to reduce to the following statement: for each $\underline{x}$ and each indecomposable idempotent $e \in \mathrm{End}(\underline{x})$, the corresponding object V in $\mathbf{Kar}(S\mathcal{TL})$ is isomorphic to $V_{\underline{y}}$ for some $\underline{y}$ with $\ell(\underline{y}) \leq \ell(\underline{x})$, with equality if and only if $e = \mathbb{1}$ modulo $I_{<\underline{x}}$.

Any diagram in $CM(\underline{x}, \underline{x})$ factors as $S \circ T$, for some triple $(\underline{z}, S, T)$ where $\ell(\underline{z}) \leq \ell(\underline{x})$, $T \in \mathbb{C}(\underline{x}, \underline{z})$ is a cap diagram, and $S \in \mathbb{C}(\underline{z}, \underline{x})$ is a cup diagram (see [25, section 2] for the definition of cap and cup diagrams). Then one can expand e in the diagram basis

$$e = \sum_{\underline{z}} \sum_{(\underline{z},S,T)} a_{S,T} S \circ T$$

with some coefficients $a_{S,T} \in \mathbb{k}$. Choose a sequence $\underline{y}$ of maximal length such that there exists a triple $(\underline{y}, S, T)$ with $a_{S,T} \neq 0$. Note that $\underline{y} = \underline{x}$ precisely when $e = \mathbb{1}$ modulo $I_{<\underline{x}}$.

We wish to show that there is some triple $(\underline{y}, X, Y)$ such that

$$Y \circ e \circ X \in \mathbb{k}^{\times} \subset \mathbb{k} = \mathrm{End}(\underline{y})/I_{<\underline{y}}.$$

Let $I_{\not\geq \underline{y}}$ denote the ideal of all morphisms which factor through any sequence shorter than $\underline{y}$, or any other sequence of the same length. We now proceed to work in the quotient category $\mathcal{STL}/I_{\not\geq \underline{y}}$. The image of e is still a nonzero idempotent, expanded as above except only using triples with $\underline{z} = \underline{y}$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{k}$. Suppose that

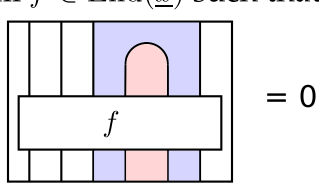
$$T \circ e \circ S \in \mathfrak{m} \subset \mathbb{k} = \mathrm{End}(\underline{y})/I_{<\underline{y}}$$

for all triples $(\underline{y}, S, T)$. By expanding $e^3 = e$ one can deduce that each $a_{S,T} \in \mathfrak{m}$. But this is a contradiction, as $\mathfrak{m}\mathrm{End}(\underline{x})$ is contained in the Jacobson radical of $\mathrm{End}(\underline{x})$, and no non-zero idempotent can be contained in the Jacobson radical.

The map $Y \circ e$ induces a map $V \to \underline{y}$, and $e \circ X$ induces a map in the other direction. By composing these further with the chosen idempotent e_0 inside $\underline{y}$, we obtain maps $e_0 \circ Y \circ e \colon V \to V_{\underline{y}}$ and $e \circ X \circ e_0 \colon V_{\underline{y}} \to V$. Composing these maps we get an endomorphism of $V_{\underline{y}}$ which projects to an invertible map in $\mathrm{End}(V_{\underline{y}})/I_{<\underline{y}} = \mathbb{k}$, so that it must be invertible in the local ring $\mathrm{End}(V_{\underline{y}})$. Therefore, $V_{\underline{y}}$ occurs as a summand of V, and since V is indecomposable, we have $V \cong V_{\underline{y}}$. □

2.3. **Orthogonality.** In the rest of this chapter, we discuss the case when e_0 has an alternative description as the unique idempotent perpendicular to $I_{<\underline{x}}$. In particular, e_0 is canonically defined, and $V_{\underline{x}}$ is well-defined up to unique isomorphism. In this case, the recursive formula of the following section will make the fact that all other idempotents factor through shorter expressions immediately obvious.

Let $T = T(\underline{x}) \subset \mathrm{End}(\underline{x})$ be the right perpendicular space to $I_{<\underline{x}}$. In other words, T is the $\mathbb{k}$-module consisting of all $f \in \mathrm{End}(\underline{x})$ such that $cf = 0$ for any cap c.



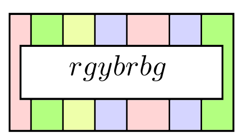
(The lack of color in some regions is supposed to represent the irrelevance of those colors.) Similarly, let $B = B(\underline{x}) \subset \mathrm{End}(\underline{x})$ be the left perpendicular space, the $\Bbbk$-module consisting of all $f \in \mathrm{End}(\underline{x})$ such that $fc = 0$ for any cup c. Clearly $\iota(B) = T$.

Claim 2.14. *Suppose that $B(\underline{x})$ contains an element f for which the coefficient of the identity is invertible in $\Bbbk$. Then $B = T$ and both are spanned by f. Every element of B is fixed by ι. Moreover, B contains a unique idempotent $JW(\underline{x})$, which is determined within B by the fact that the coefficient of the identity is 1. The idempotent $JW(\underline{x})$ is indecomposable and central, and it is the unique indecomposable idempotent not contained in $I_{<\underline{x}}$.*

Proof. Let us write $f = \lambda \mathbb{1} + f'$, where $f' \in I_{<\underline{x}}$ and $\lambda \in \Bbbk$ is invertible. For any $g \in T$ one has $f'g = 0$, since any non-identity diagram has a cap on bottom. Therefore $fg = \lambda g$. By the same token, if $g \in \mu\mathbb{1} + I_{<\underline{x}}$ then $fg = \mu f$. In particular, $g = \mu\lambda^{-1} f$. This proves that every element of T is in the $\Bbbk$-span of f. By the same token, every element of B is in the $\Bbbk$-span of $\iota(f)$. Thus $\iota(f) \in T$ is a multiple of f, and the coefficient of the identity is also λ, so that $\iota(f) = f$. Therefore $B = T = \Bbbk \cdot f$, and every element is ι-fixed. Moreover, letting $JW(\underline{x}) = \lambda^{-1} f$, the above argument proves that $JW(\underline{x})^2 = JW(\underline{x})$.

If $g \in \mathrm{End}(\underline{x})$ is any element, and $g \in \mu\mathbb{1} + I_{<\underline{x}}$, then $JW(\underline{x})g = gJW(\underline{x}) = \mu JW(\underline{x})$, so that $JW(\underline{x})$ is central. In particular, the ideal of $JW(\underline{x})$ is free of rank 1, from which it follows that the idempotent is indecomposable. Moreover, $JW(\underline{x})g = 0$ if and only if $g \in I_{<\underline{x}}$, so that every other indecomposable idempotent is contained in $I_{<\underline{x}}$. □

This idempotent $JW(\underline{x})$, which we call the *top idempotent*, is akin to the Jones-Wenzl projectors defined for usual Temperley-Lieb algebras. We draw $JW(\underline{x})$ as a box labeled by $\underline{x}$, as in this example.



It is possible that B does not contain any element with invertible coefficient of the identity, in which case we say that the top idempotent does not exist. In this case, the special idempotent e_0 discussed above can be more complicated, and will not be orthogonal to $I_{<\underline{x}}$.

2.4. **A recursive formula for top idempotents.** We use quantum number notation to indicate certain elements in $\Bbbk$. Let $[2] \in \Bbbk$ be the image of δ, and let $[1] = 1$ and $[0] = 0$. One defines the quantum number $[m] \in \Bbbk$ for $m \in \mathbb{Z}$ by the recursive formula

$$[2][m] = [m+1] + [m-1]. \tag{2.3}$$

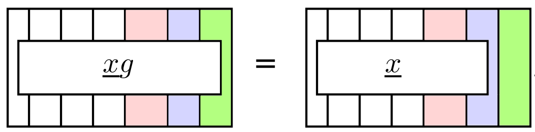
Given a sequence of colors $\underline{x}$, a *subsequence* $\underline{y} \subset \underline{x}$ will always indicate a consecutive subsequence. A subsequence is *alternating* if it alternates between two colors in S; it is *maximal alternating* if it can not be extended to a longer alternating subsequence. An *initial subsequence* is a sequence consisting of the first k colors in $\underline{x}$, and a *final subsequence* is a sequence consisting of the last k colors in $\underline{x}$, for any $1 \leq k \leq \ell(x)$.

The *tail* of $\underline{x}$ is the maximal alternating final subsequence. For example, the tail of $gbryrgbrbrb$ is the final 5 colors $brbrb$.

In this section we provide a recursive formula for top idempotents under the assumption that certain quantum numbers are invertible in $\Bbbk$. This imitates a formula from [24], giving the Jones-Wenzl projectors in the usual Temperley-Lieb category.

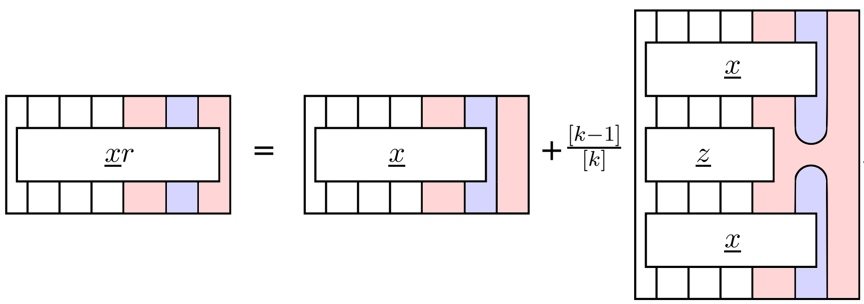
Proposition 2.15. *When $\ell(\underline{x}) \leq 2$, the idempotent $JW(\underline{x})$ exists, and is equal to the identity map. Now suppose that $\underline{x} = \ldots rb$ and that $JW(\underline{y})$ exists for all initial subsequences of $\underline{x}$. Extending $\underline{x}$ by a color $g \neq r, b$ one has*

(2.4)



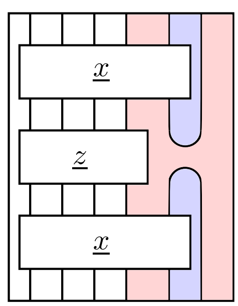
Extending $\underline{x}$ by r, when $[k]$ is invertible one has

(2.5)



The sequence $\underline{z} = \ldots r$ is the initial subsequence of $\underline{x}$ which is only missing the final b. The number k appearing in (2.5) *is the length of the tail of $\underline{x}$. Moreover, the map*

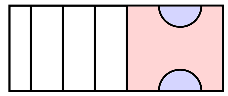
(2.6) $-\frac{[k-1]}{[k]}$



is an idempotent in $\mathrm{End}(\underline{x}r)$ orthogonal to $JW(\underline{x}r)$. If $[k]$ is not invertible then $JW(\underline{x}r)$ does not exist.

Note that the idempotent $JW(\underline{z})$ which appears in (2.6) can be replaced by the identity map of $\underline{z}$. After all, any non-identity term will have a cup on top, which will annihilate $JW(\underline{x})$. We included the idempotent $JW(\underline{z})$ because it implies that the idempotent (2.6) factors through $V_{\underline{z}}$ inside $\underline{z}$.

Exercise 2.16. When (2.5) holds, show that the coefficient of

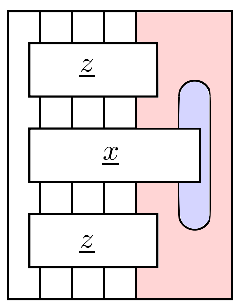
(2.7) 

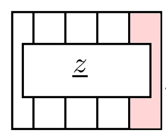
in $JW(\underline{x}r)$ is precisely $\frac{[k-1]}{[k]}$. Hint: replace each $JW(\underline{x})$ with a linear combination of crossingless matchings in the RHS of (2.5), and observe that only a single term could possibly contribute to this coefficient.

Proof. When $\ell(\underline{x}) \leq 2$, $CM(\underline{x}, \underline{x})$ only contains the identity map, and $I_{<\underline{x}} = 0$. It is clear that $B = T = \mathrm{End}(\underline{x})$ and $JW(\underline{x}) = \mathbb{1}_{\underline{x}}$. We assume henceforth that $\ell(\underline{x}) \geq 2$, and that $JW(\underline{y})$ exists for all initial subsequences $\underline{y}$ of $\underline{x}$. If k is the length of the tail of $\underline{x}$, then our inductive hypothesis implies that $[l]$ is invertible for all $l < k$.

Suppose that $g \neq r, b$. Any non-identity diagram in $CM(\underline{x}g, \underline{x}g)$ must begin with a cup, and it is inconsistent with the coloring for this cup to involve the final strand. Therefore any non-identity diagram will kill the RHS of (2.4), because a cup enters $JW(\underline{x})$. The RHS of (2.4) is clearly in $B(\underline{x}g)$, and the coefficient of the identity is equal to 1 since this is true also in $JW(\underline{x})$, so that the RHS is equal to $JW(\underline{x}g)$.

Now we extend $\underline{x}$ by r. We claim that

(2.8) 

$= -\frac{[k]}{[k-1]}$ 

.

To show this we use induction, assuming that $JW(\underline{x})$ was defined using either (2.4) or (2.5) to extend $\underline{z}$ by b. When $k = 2$, $JW(\underline{x})$ is defined using (2.4), and (2.8) is clear since the value of a circle is $-[2]$. When $k > 2$, $JW(\underline{x})$ is defined using (2.5), and the size of the tail of $\underline{z}$ is $k - 1$. Writing $JW(\underline{x})$ as a linear combination of crossingless matchings, the only ones with nonzero contribution to (2.8) are the identity and the diagram in (2.7). The identity contributes $-[2]$ times $JW(\underline{z})$, and the diagram in (2.7) contributes $\frac{[k-2]}{[k-1]}$ times $JW(\underline{z})$. Adding these, one obtains $\frac{[k-2]-[2][k-1]}{[k-1]} = \frac{-[k]}{[k-1]}$ times $JW(\underline{z})$, as desired.

Suppose that $[k]$ is invertible. The RHS of (2.5) is obviously killed by any cup other than a cup on the final subsequence rbr. This final cup also kills the RHS, by (2.8). The coefficient of the identity in the RHS is only affected by the first term, and is therefore equal to 1. Thus the RHS of (2.5) is by definition equal to $JW(\underline{x}r)$. The statement about the orthogonal idempotent is also clear.

Now suppose that $[k]$ is not invertible. Multiplying the RHS of (2.5) by $[k]$, one obtains a map in B which is well-defined. The coefficient of (2.7) is now $[k-1]$, which is invertible, but the coefficient of the identity is $[k]$, which is not invertible. If

$JW(\underline{x}r)$ exists then any element of B is $JW(\underline{x}r)$ multiplied by the coefficient of the identity; if the coefficient of the identity is non-invertible, then every coefficient is non-invertible. This is a contradiction, so that $JW(\underline{x}r)$ can not exist. □

Remark 2.17. One could also prove this recursive formula using the usual recursive formula from [24] for Jones-Wenzl projectors, combined with Proposition 2.22 below. However, we felt this proof was still useful and motivational.

Corollary 2.18. *Suppose that all quantum numbers are invertible. Let $V_{\underline{x}}$ denote the image of $JW(\underline{x})$, an indecomposable object of the Karoubi envelope* **Kar**$(\mathcal{STL})$. *If $\underline{x}$ ends in rb, and $\underline{z}$ is the initial sequence missing only the final b, then one has*

$$V_{\underline{x}}V_{bg} \cong V_{\underline{x}g}, \tag{2.9}$$

$$V_{\underline{x}}V_{br} \cong V_{\underline{x}r} \oplus V_{\underline{z}}. \tag{2.10}$$

Proof. This is implied by (2.4) and (2.5), which give a decomposition of the identity of $V_{\underline{x}}V_{bg}$ and $V_{\underline{x}}V_{br}$ respectively into orthogonal idempotents which factor through the appropriate objects. □

2.5. A descriptive formula for top idempotents.

The recursive formula of Proposition 2.15 does not completely answer the question of when the map $JW(\underline{x})$ exists. After all, it is possible for $JW(\underline{x})$ to exist even when $JW(\underline{y})$ does not exist for an initial subsequence $\underline{y} \subset \underline{x}$. As an example, consider the case when $\underline{x}$ is an alternating sequence, so that the question reduces to the usual Temperley-Lieb algebra and its Jones-Wenzl projectors.

Claim 2.19. *Let $\underline{x} = rbrb\ldots$ be an alternating sequence of length $k+1$. Then $JW(\underline{x})$ exists if and only if the quantum binomial coefficients $\begin{bmatrix} k \\ m \end{bmatrix}$ are invertible in $\Bbbk$, for all $0 \le m \le k$. Equivalently, the Jones-Wenzl projector in the usual Temperley-Lieb algebra (on k strands) exists if and only if the quantum binomial coefficients are invertible.*

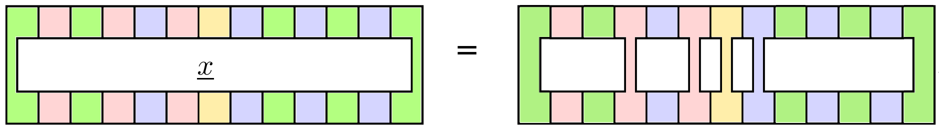
This claim is fundamental to the theory of Temperley-Lieb algebras, but we have not been able to find it in the literature. A proof by Ben Webster can be found in the appendix.

Remark 2.20. It was shown by Westbury [25, Lemma 5] that, when all the quantum numbers $[m]$ are invertible for $0 \le m \le k$, then the Temperley-Lieb algebra is semisimple. In other words, when these quantum numbers are invertible, then the Temperley-Lieb algebra has many idempotents, all the idempotents one can generically expect. The Jones-Wenzl projector is just one of these idempotents, and its existence is a weaker condition.

Example 2.21. $JW(rbrb)$ exists when $[3]$ is invertible. This can happen even when $[2]$ is not invertible (e.g., when $[2] = 0$ one has $[3] = -1$), in which case $JW(rbr)$ does not exist.

Now we use Claim 2.19 to give an exact condition for whether $JW(\underline{x})$ exists.

Proposition 2.22. *Suppose that $\begin{bmatrix} k \\ m \end{bmatrix}$ is invertible whenever $0 \leq m \leq k$ and $k+1$ is the length of a maximal alternating subsequence of $\underline{x}$. Then one has*

(2.11) 

In this equation, the smaller rectangles which appear are the JW maps associated to maximal alternating subsequences. On the other hand, if some such $\begin{bmatrix} k \\ m \end{bmatrix}$ is not invertible, then $JW(\underline{x})$ does not exist.

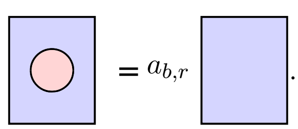
This result should hardly be unexpected in light of Proposition 2.15, as this is exactly the morphism which the recursive formula of the previous section would construct.

Proof. Under the assumptions of invertibility, clearly the RHS of (2.11) exists, and clearly it satisfies the defining conditions of $JW(\underline{x})$.

Now suppose that some $\begin{bmatrix} k \\ m \end{bmatrix}$ is not invertible, where $k+1$ is the length of a maximal alternating subsequence $\underline{y}$. By the algebraic proof of Claim 2.19, there is some element f of $B(\underline{y})$ with a noninvertible coefficient of the identity, but with an invertible coefficient of some non-identity diagram D. Taking the horizontal concatenation of f with $JW(\underline{z})$ for the other maximal alternating subsequences $\underline{z}$ of $\underline{x}$, one obtains an element of $B(\underline{x})$ with a noninvertible coefficient of the identity, but with an invertible coefficient of $\mathbb{1} \otimes D \otimes \mathbb{1}$. This contradicts the existence of $JW(\underline{x})$, using the same argument as in the end of the proof of Proposition 2.15. □

2.6. **Generalizations.** We no longer assume that $\Bbbk$ is a $\mathbb{Z}[\delta]$-algebra. Let $A = (a_{s,t})$ be an $S \times S$ matrix with values in $\Bbbk$. We will only be interested in the values of $a_{s,t}$ for $s \neq t$. By convention, we set $a_{s,s} = 2$, so that A is a Cartan matrix (in the sense of the next chapter).

Definition 2.23. Let $\mathcal{STL}(A)$ be the 2-category defined as in Definition 2.1 except that instead of (2.2) one has

(2.12) 

In other words, a circle still evaluates to a scalar, but which scalar depends on the color both inside and outside. When $\Bbbk$ is a $\mathbb{Z}[\delta]$-algebra, the special matrix with $a_{s,t} = -\delta$ for all $s \neq t$ will recover the original 2-category $\mathcal{STL}$.

For any two fixed colors $s \neq t \in S$, there is a notion of *two-colored quantum numbers* $[m]_{s,t}$. These are defined by recursive formulae, starting with $[0]_{s,t} = 0$ and $[1]_{s,t} = 1$. Then one has

$$[2]_{s,t} = -a_{s,t}, \qquad [2]_{t,s} = -a_{t,s}, \tag{2.13}$$

(2.14) $[2]_{s,t}[m]_{t,s} = [m-1]_{s,t} + [m+1]_{s,t}, \qquad [2]_{t,s}[m]_{s,t} = [m-1]_{t,s} + [m+1]_{t,s}.$

It is not difficult to see that $[m]_{s,t} = [m]_{t,s}$ when m is odd, and that when m is even, $[m]_{s,t}$ is a multiple of $[2]_{s,t}$ and $\frac{[m]_{s,t}}{[2]_{s,t}} = \frac{[m]_{t,s}}{[2]_{t,s}}$. There are many analogies between these two-colored quantum numbers and usual quantum numbers; see [5, Appendix] for more details.

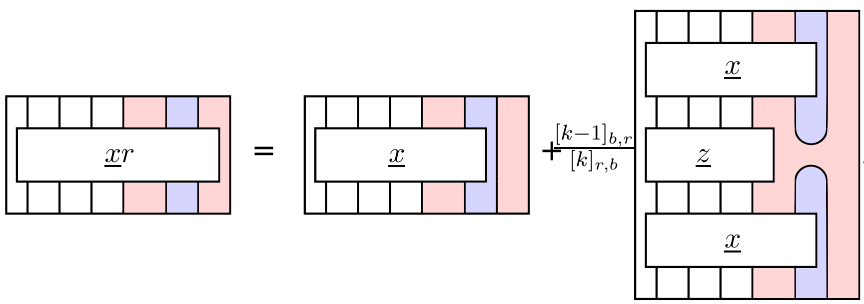
The two-colored quantum binomial coefficients $\begin{bmatrix} k \\ m \end{bmatrix}_{s,t}$ are defined by the formula

$$\begin{bmatrix} k \\ m \end{bmatrix}_{s,t} = \frac{[1]_{s,t}[2]_{s,t}\cdots[k]_{s,t}}{[1]_{s,t}\cdots[m]_{s,t}[1]_{s,t}\cdots[k-m]_{s,t}}. \tag{2.15}$$

By comparing the numbers of even and odd quantum numbers, it is easy to see that $\begin{bmatrix} k \\ m \end{bmatrix}_{s,t} = \begin{bmatrix} k \\ m \end{bmatrix}_{t,s}$ unless k is even and m is odd, in which case $\begin{bmatrix} k \\ m \end{bmatrix}_{s,t}$ is a multiple of $[2]_{s,t}$ and $\frac{\begin{bmatrix} k \\ m \end{bmatrix}_{s,t}}{[2]_{s,t}} = \frac{\begin{bmatrix} k \\ m \end{bmatrix}_{t,s}}{[2]_{t,s}}$.

The results of Claim 2.14 still hold. Proposition 2.15 and its corollaries will still hold, after replacing quantum numbers with the two-colored quantum numbers. More precisely, one should replace (2.5) with

(2.16)



Corollary 2.18 holds once one replaces the first sentence with "Suppose all two-colored quantum numbers are invertible, for all pairs $s \neq t \in S$." The analogue of Proposition 2.22 is:

Corollary 2.24. *Suppose that $\begin{bmatrix} k \\ m \end{bmatrix}_{s,t}$ is invertible whenever $0 \le m \le k$ and there exists a maximal alternating subsequence $\ldots ts \subset \underline{x}$ of length $k+1$. Then $JW(\underline{x})$ exists and* (2.11) *holds. If one of these two-colored quantum binomial coefficients is not invertible, then $JW(\underline{x})$ does not exist.*

3. Universal Soergel bimodules

3.1. **Universal Coxeter groups and Hecke algebras.** The *universal Coxeter group* associated to a finite set S is a group W with a Coxeter presentation having generators S and relations $s^2 = 1$ for each $s \in S$. An *expression* is a sequence $\underline{x} = s_1 s_2 \ldots s_d$ of elements of S; it has length d and is *reduced* if $s_i \neq s_{i+1}$ for any i. Removing the underline, we let x denote the corresponding product of generators in W. Each element of W has a unique reduced expression.

The *Hecke algebra* $\mathbf{H}$ of W is the $\mathbb{Z}[v, v^{-1}]$-algebra having a presentation with generators H_s for $s \in S$, and relations

$$(H_s + v)(H_s - v^{-1}) = 0 \tag{3.1}$$

for each $s \in S$. It has a *Kazhdan-Lusztig* or *KL basis* $\{b_w\}_{w \in W}$ as a $\mathbb{Z}[v, v^{-1}]$-module. One has $b_1 = 1$ and $b_s = H_s + v$ for each $s \in S$. Dyer [4, Lemma 6.1] has shown that the KL basis is given by the following recursive formula.

Proposition 3.1. *(The Dyer formula) If $\underline{x} = s_1 s_2 \dots rb$ is a reduced expression, $g \in S$, $g \neq r$ and $g \neq b$, then*

$$b_x b_g = b_{xg}. \tag{3.2}$$

On the other hand,

$$b_x b_r = b_{xr} + b_{xb}. \tag{3.3}$$

Note that the reduced expression for xb is the initial subsequence $\underline{z}$ of $\underline{x}$ which is missing only the final b.

Whenever $\underline{x}$ is a reduced expression but $\underline{x}s$ is not, one can show that

$$b_x b_s = (v + v^{-1}) b_x. \tag{3.4}$$

Between this equation and the Dyer formula, one can compute the product of any b_w for $w \in W$ with any b_s for $s \in S$.

3.2. Realizations.

Definition 3.2. A *realization* of a universal Coxeter group W over $\Bbbk$ is a free, finite rank $\Bbbk$-module $\mathfrak{h}$ together with its dual $\mathfrak{h}^*$, a choice of *simple roots* $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ and a choice of *simple coroots* $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$, satisfying $\langle \alpha_s, \alpha_s^\vee \rangle = 2 \in \Bbbk$.

The *Cartan matrix* attached to a realization is the $S \times S$ matrix $A = (a_{s,t})$ with values in $\Bbbk$, given by $a_{s,t} = \langle \alpha_t, \alpha_s^\vee \rangle$. We do not assume that the simple roots span $\mathfrak{h}^*$ or that the simple coroots span $\mathfrak{h}$, so that A need not determine the realization.

We assume in this paper that our realization satisfies *Demazure surjectivity*, which is the condition that $\langle \alpha_s, \cdot \rangle$ and $\langle \cdot, \alpha_s^\vee \rangle$ are both surjective maps to $\Bbbk$.

Given any realization, there is an action of W on $\mathfrak{h}$ defined on generators by the formula $s(v) = v - \langle \alpha_s, v \rangle \alpha_s^\vee$. The contragredient action on $\mathfrak{h}^*$ is given by $s(f) = f - \langle f, \alpha_s^\vee \rangle \alpha_s$. Let R be the coordinate ring of $\mathfrak{h}$, or in other words, the $\Bbbk$-linear polynomial ring whose linear terms are $\mathfrak{h}^*$. We give R a grading, so that $\deg(\mathfrak{h}^*) = 2$. The commutative ring R has a natural homogeneous action of W.

There is a Demazure map $\partial_s \colon R \to R^s$, defined by the formula

$$\partial_s = \frac{f - s(f)}{\alpha_s},$$

whose image is the set of s-invariant polynomials. On linear polynomials $f \in \mathfrak{h}^* \subset R$, one has $\partial_s(f) = \langle f, \alpha_s^\vee \rangle \in \Bbbk$.

3.3. **Diagrammatics for Soergel bimodules.** Instead of defining and developing the theory of Soergel bimodules, we prefer to follow the diagrammatic approach developed in [8]. Our object of study will be a certain monoidal category with graded Hom spaces, given diagrammatically.

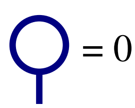
Definition 3.3. A *Soergel graph* is a certain kind of finite graph embedded in the planar strip $\mathbb{R} \times [0,1]$. The edges in this graph are colored by $s \in S$. The only vertices allowed are trivalent vertices connecting three edges of the same color, and univalent vertices (called *dots*). We also allow edges which meet no vertices, forming a circle. This graph is allowed to have a boundary on the walls of the strip (i.e. edges may terminate at $\mathbb{R} \times \{0\}$ or $\mathbb{R} \times \{1\}$, though these termination points are not counted as vertices). The edge labels that meet the boundary give two sequences of colors, the *bottom boundary* and the *top boundary*. A *region* of the graph is a connected component of the complement of the Soergel graph in $\mathbb{R} \times [0,1]$. Finally, we may place a homogeneous polynomial in R inside each region of the graph. We consider these graphs up to isotopy, though this isotopy must preserve $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. A Soergel graph has a *degree*, which accumulates $+1$ for every dot, -1 for every trivalent vertex, and the degree of each polynomial.

In particular, the connected components of a Soergel graph have a single color. For numerous examples, look ahead.

Definition 3.4. Let $\mathcal{D}$ be the monoidal category defined as follows. The objects are monoidally generated by $s \in S$, so that a general object is an expression $\underline{x}$ (not necessarily reduced, possibly empty). Given two expressions $\underline{x}$ and $\underline{y}$, the morphism space $\mathrm{Hom}(\underline{x}, \underline{y})$ will be the $\Bbbk$-module spanned by Soergel graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo the local relations below. This morphism space is graded by the degree of the Soergel graphs, and all the relations below are homogeneous.

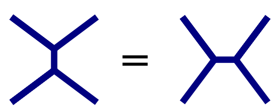
The **Needle relation**:

(3.5)

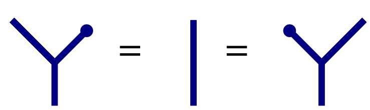


The **Frobenius relations**:

(3.6a)

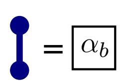


(3.6b)

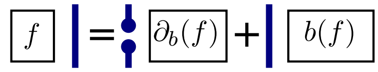


The **Barbell relation**:

(3.7)



The **Polynomial forcing relation**:

(3.8) 

This ends the definition.

We may also consider Soergel graphs on the planar disk; these have a single boundary sequence $\underline{x}$, which is to be considered only up to cyclic permutation. A Soergel graph on the disk does not represent a morphism in $\mathcal{D}$. However, as the relations above are local, one can apply them to any disk within the planar strip, so disk diagrams are useful for local calculations.

Let $\mathbf{Kar}(\mathcal{D})$ denote the Karoubi envelope of the graded, additive closure of $\mathcal{D}$. The following theorem is proven in [8], whose analogue for Soergel bimodules is known as the *Soergel Categorification Theorem*.

Theorem 3.5. *The indecomposable objects in $\mathbf{Kar}(\mathcal{D})$, up to isomorphism and grading shift, can be labeled by $w \in W$. The indecomposable object B_w is the unique summand inside $\underline{w}$ for a reduced expression of w which is not a direct summand of $\underline{y}$ for any shorter expression. There is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras*

$$\mathbf{H} \to [\mathbf{Kar}(\mathcal{D})]$$

from the Hecke algebra to the Grothendieck ring of $\mathbf{Kar}(\mathcal{D})$, which sends b_s to the symbol of the generating object s, which is equal to B_s.

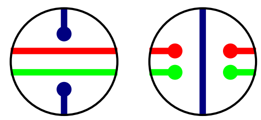
Soergel conjectured that, when $\Bbbk$ has characteristic zero and the representation $\mathfrak{h}$ is "reflection-faithful," this isomorphism sends b_w to $[B_w]$. Our goal in the rest of this paper is to give a criterion for when $b_w \mapsto [B_w]$, which will happen when we can categorify the Dyer formula.

3.4. **Maximally connected graphs and minimal degrees.** This section is an adaptation of [5, section 5.3.3], which only treated the case of two colors. However, the arguments are almost identical.

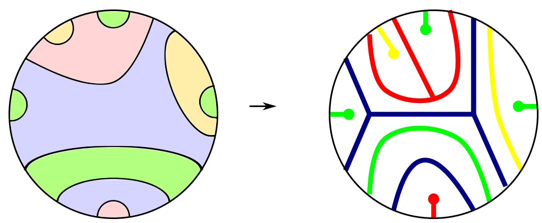
Using the relations in Definition 3.4, it is not hard to show that every graph is in the span of a graph containing only *simple trees* with polynomials. In other words, each connected component of the graph is a tree with non-empty boundary. Any two trees with the same boundary are equal by (3.6a) and (3.6b). Moreover, this tree contains a dot precisely when the boundary is a single point, in which case the tree has no trivalent vertices; we call such a tree a *boundary dot*. Moreover, one can assume that there is a single polynomial, and it occurs in a region of one's choosing (say, the leftmost region). The proof in [5, Proposition 5.19] works verbatim.

Definition 3.6. A Soergel graph containing only simple trees is *maximally connected* if it has no polynomials, and satisfies the following condition for each $s \in S$. Consider the subgraph Γ consisting of all the edges colored by $S \setminus s$. Then Γ splits the planar strip into regions, and each region may contain at most one connected component colored s.

It is easy to see that maximally connected Soergel graphs with a given boundary exist, and that for any graph which is not maximally connected, one can produce a maximally connected graph of smaller degree by "fusing" two edges (see [5, section 5.3.3]) or removing a polynomial. However, not all maximally connected graphs have the same degree, as the following examples show.



Given a S-colored crossingless matching on the planar disk (with at least two colors), one can obtain a maximally connected Soergel graph by taking a deformation retract of each colored region. A quick inductive argument shows that any such Soergel graph has degree $+2$. The choice of deformation retract is irrelevant, because any two trees with the same boundary are equal.



The regions in the resulting graph (RHS) correspond to the strands in the original crossingless matching (LHS). Therefore, each region is bounded by exactly two colors, and meets the boundary exactly twice. It is easy to recover the colored crossingless matching from the resulting graph: simply deformation retract each region into a strand, and use the colors of the graph to color the regions between strands.

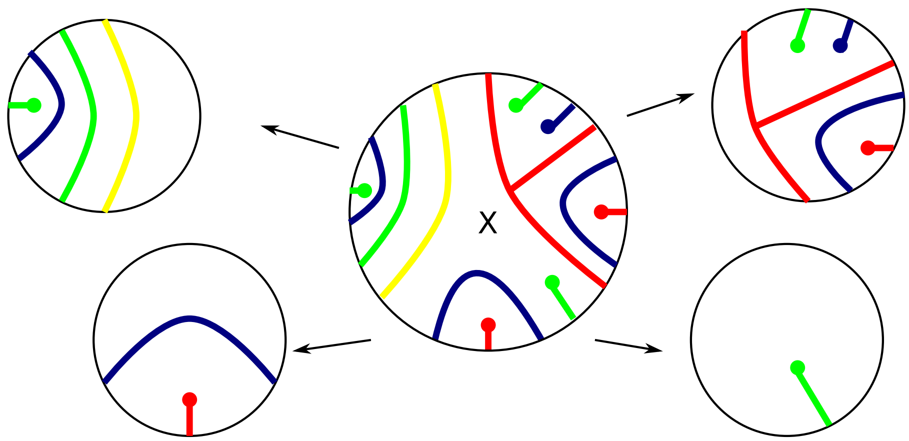
Proposition 3.7. *Let $\underline{x} = s_1 s_2 \dots s_d$ be a sequence representing the boundary of a Soergel graph on the planar disk (so that we only consider $\underline{x}$ up to rotation). Then any maximally connected Soergel graph on the disk with boundary $\underline{x}$ has degree $\geq 2 - m$, where m is the number of repetitions in $\underline{x}$ (i.e. the number of $1 \leq i \leq d$ such that $s_i = s_{i+1}$, where we set $s_{d+1} = s_1$). If $\underline{x}$ has no repetitions and the graph has degree 2, then it arose as the deformation retract of some colored crossingless matching.*

Proof. It is easy to reduce to the case where $\underline{x}$ has no repetitions.

The maximally connected graph splits the disk into regions. Since there are no cycles, each region must meet the boundary of the disk at least once, say between s_i and s_{i+1}. Suppose that a region X meets the boundary of the disk k times, so that there are distinct indices $i_1, i_2, \dots, i_k$ such that X meets the boundary between s_{i_j} and s_{i_j+1}. By following the walls of X we see that the colors s_{i_j+1} and $s_{i_{j+1}}$ are equal, and the maximally connected condition implies that the k colors s_{i_j} are all distinct. Therefore, the number of bordering colors of a region is equal to the number of times that region meets the boundary. This number is always at least 2, since there are no repetitions in $\underline{x}$.

If each region meets the boundary exactly twice, then we can deformation retract the regions into strands to obtain a colored crossingless matching, as above. It is easy to see that this yields a bijection between S-colored crossingless matchings, and maximally connected graphs where each region meets the boundary exactly twice.

If a region meets the boundary k times, one can use this region to cut the overall graph into k subgraphs, each of which is either a boundary dot, or a maximally connected graph with one repetition. This is illustrated in the picture below, for a region X meeting the boundary 4 times. Using induction therefore, each subgraph has degree at least $+1$. If any region meets the boundary ≥ 3 times, it follows that the overall degree of the original graph is ≥ 3. $\square$

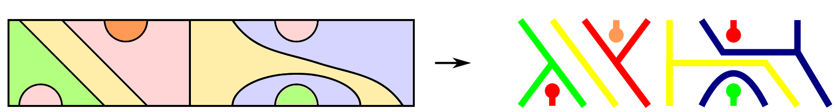


Proposition 3.7 allows one to place a lower bound on the degree of the Hom space between two objects in $\mathcal{D}$.

Corollary 3.8. *Let $\underline{x} = s_1 \dots s_d$ and $\underline{y} = t_1 \dots t_k$ be two nonempty reduced expressions. If $s_1 = t_1$ and $s_d = t_k$ then every nonzero morphism in $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$ has degree ≥ 0. Otherwise, every nonzero morphism has degree ≥ 1. Similarly, nonzero morphisms in $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \emptyset)$ and $\mathrm{Hom}_{\mathcal{D}}(\emptyset, \underline{y})$ have degree ≥ 1, while every nonzero morphism in $\mathrm{Hom}_{\mathcal{D}}(\emptyset, \emptyset)$ has degree ≥ 0.*

Proof. Let $\underline{y}^{\mathrm{op}}$ denote the sequence $\underline{y}$ in reverse. Viewing $\underline{x}(\underline{y}^{\mathrm{op}})$ as a long sequence on the circle, there is one repetition if $s_1 = t_1$, and one repetition if $s_d = t_k$. By Proposition 3.7, the minimal degree of any map is at least 2 minus the number of repetitions. Similarly, if $\underline{y}$ is empty, then $\underline{x}$ has at most one repetition, if $s_1 = s_d$. $\square$

Given an S-colored crossingless matching on the planar strip, one can obtain a maximally connected Soergel graph of degree 0 by deformation retract, as in the example below.



Corollary 3.9. *Every degree zero map between nonempty reduced expressions in $\mathcal{D}$ arises from an S-colored crossingless matching on the planar strip.*

The lower bound in Corollary 3.8 can also be obtained from Soergel's Hom formula. The advantage of this approach is an explicit description of the morphisms in the lowest degree. This description can also be obtained, with some work, from the second author's light leaves basis for Hom spaces.

3.5. **The main theorem.**

Proposition 3.10. *Let A be the Cartan matrix of the realization. There is a non-monoidal functor from $\mathcal{STL}(A)$ (viewed as a 1-category) to $\mathcal{D}$, which sends the 1-morphism $\underline{x}$ in $\mathcal{STL}(A)$ to the object $\underline{x}$ in $\mathcal{D}$, and sends a 2-morphism in $\mathcal{STL}(A)$ corresponding to an S-colored crossingless matching to the corresponding degree 0 deformation retract. This functor is essentially surjective (in the Karoubi envelope) and fully faithful onto maps of degree 0.*

Proof. First we must show that this deformation retract map preserves the algebra structure. It is easy to check (2.1). Relation (2.12) follows from $\partial_b(\alpha_r) = a_{b,r}$, a deduction we leave as an exercise to the reader. Thus the functor is well-defined. Corollary 3.9 implies that it is full onto degree 0. Soergel's Hom formula implies that the dimensions of Hom spaces agree, and thus it is also faithful. By the classification of Theorem 3.5, each indecomposable object in **Kar**$(\mathcal{D})$ appears as a summand of an object in the image of the functor, so the functor is essentially surjective. □

This non-monoidal functor can be upgraded to a genuine 2-functor to the 2-category of singular Soergel bimodules. For definitions and a proof in the two-color case, see [5, section 5.2.3].

By Proposition 3.10, every idempotent in $\mathrm{End}_{\mathcal{D}}(\underline{x})$ for a nontrivial reduced expression comes from an idempotent in $\mathrm{End}_{\mathcal{STL}(A)}(\underline{x})$. In particular, the theory of top idempotents implies that, when $JW(\underline{x})$ exists, it must project to the indecomposable object $B_x \in$ **Kar**$(\mathcal{D})$. Our main theorem is now immediately implied by Corollary 2.18.

Theorem 3.11. *Suppose that all two-colored quantum numbers are invertible in $\Bbbk$. Then for a reduced expression $\underline{x} = \ldots rb$ and a simple reflection $s \in S$ one has the following isomorphism in **Kar**$(\mathcal{D})$, which categorifies the Dyer formula. As a consequence, the map $\mathbf{H} \to [\mathbf{Kar}(\mathcal{D})]$ sends $b_w \mapsto [B_w]$.*

$$B_x B_s \cong \begin{cases} B_x(1) \oplus B_x(-1) & \text{if } s = b, \\ B_{xs} & \text{if } s \neq r, b, \\ B_{xr} \oplus B_{xb} & \text{if } s = r. \end{cases} \tag{3.9}$$

Proof. The case when $s = b$ follows from the case when $x = s$, and was proven in [8]. The other cases follow from the recursion formula for top idempotents. □

Moreover, when certain two-colored quantum binomial coefficients vanish, it is immediately clear for which w the statement that $b_w \mapsto [B_w]$ will fail, as in Corollary 2.24.

Remark 3.12. The vanishing of two-colored quantum numbers determines which Coxeter quotient of W acts faithfully on $\mathfrak{h}$, as was discussed in [5, Appendix]. In particular, if the universal Coxeter group W acts faithfully on $\mathfrak{h}$, then all two-colored quantum numbers are nonzero. Therefore, $b_w \mapsto [B_w]$ is always satisfied for a faithful realization over a field $\Bbbk$.

Remark 3.13. The typical crystallographic setting for universal Coxeter groups is the realization where $\Bbbk = \mathbb{Z}$ and $a_{s,t} = -2$ for all $s \neq t$. In this case, the two-colored quantum number $[m]_{s,t}$ is equal to the integer m. Specializing to a field of finite characteristic, it is clear which binomial coefficients vanish.

Remark 3.14. It is not difficult, using Soergel's Hom Formula (see [8]), to extend these results to those reduced expressions $\underline{x}$ in arbitrary Coxeter groups W for which every element $y \leq x$ in W has a unique reduced expression. For general Coxeter groups, morphisms are not spanned by univalent-trivalent Soergel graphs as above, requiring a more complicated definition of Soergel graphs. However, the Hom formula implies that univalent-trivalent graphs are sufficient for these kinds of expressions.

Remark 3.15. In unpublished work, the second author proved Soergel's conjecture for universal Coxeter groups when $\Bbbk = \mathbb{R}$, while considering the wider study of "large" Coxeter groups. The proof involved singular Soergel bimodules, and produced indecomposable bimodules using a formula analogous to (2.11). For more details, contact the second author.

The isomorphism (3.9) categorified what we have called the Dyer formula. However, Dyer [4] produces several other formulas for Hecke algebras of universal Coxeter groups. Most of these can also be deduced fairly easily from our main theorem. Let us sketch the connections here; to go into any more depth would require too much notation, but the avid reader should be able to draw the correct conclusion. We assume below that all two-colored quantum numbers are invertible.

Dyer's formula [4, (3.12)] deals with the decomposition of $B_v B_w$ into indecomposables. There are two cases. In the first case, $\ell(v) + \ell(w) = \ell(vw)$. Proposition 2.22 implies that placing JW_v next to JW_w will result in JW_{vw}, except when the tail of v overlaps with the "head" (i.e. maximal alternating initial subsequence) of w to produce a longer alternating subsequence of vw. What happens to the overlap in this case is exactly what happens in the representation theory of $\mathfrak{sl}_2$ when one takes the tensor product of two irreducible modules; it is governed by the Clebsch-Gordan formula. One can observe that Dyer's sums $C(w,i)$ are essentially a reformulation of the Clebsch-Gordan formula. In the second case, $\ell(v) + \ell(w) > \ell(vw)$, because some simple reflection s appears on the right of v and the left of w. In this case, a factor of $v + v^{-1}$ will appear in the middle, as in the decomposition $B_s B_s \cong B_s(1) \oplus B_s(-1)$. This feature of $\mathcal{D}$ has nothing to do with the multicolored Temperley-Lieb category, dealing instead with the morphisms of non-zero degree in the Soergel category, but it is amply discussed in other papers (e.g. [8]). Taking this into account, it is easy to reduce to the first case.

Dyer's formula [4, (4.1)] computes the relative KL polynomial P^y_{xw}. Let us discuss the case when $y = 1$, which gives the ordinary KL polynomial $P_{x,w}$. We assume that the reader is familiar with the light leaves basis for morphisms in $\mathcal{D}$, and the corresponding notation for subexpressions (see [8]). Let $\underline{w}$ be the unique reduced expression for w. We say that a subexpression $\mathbf{e}$ of $\underline{w}$ contains a *pitchfork* if it contains the configuration sts with $s \neq t \in S$, where t is U0 and the second s is either D1 or D0. The corresponding light leaves map contains a pitchfork, as in [8, (5.17)]. This is the morphism in $\mathcal{D}$ corresponding to a cap in the multicolored Temperley-Lieb category, and thus will be killed by precomposition with the Jones-Wenzl idempotent. We call a subexpression *pitchfork-avoiding* if it does not contain a pitchfork, and thus its light leaves map survives precomposition with the Jones-Wenzl idempotent; a standard localization argument shows that the pitchfork-avoiding light leaves maps remain linearly independent after this precomposition. It remains to observe that Dyer's set $\mathcal{P}_w(1, x)$ is precisely the set of subexpressions $\mathbf{e}$ of $\underline{w}$ which express x, and which are pitchfork-avoiding. Moreover, Dyer's number $\rho_w(\mathbf{e})$ is a computation of the defect of the light leaves map (up to an overall renormalization). The defects of these light leaves maps compute the diagrammatic character at x of the indecomposable B_w (see [8, Definition 6.23]), which agrees with the KL polynomial $P_{x,w}$ (up to renormalization).

Appendix A. The existence of Jones-Wenzl projectors (by Ben Webster)

Consider the Temperley-Lieb category $\mathcal{TL}$ over $\mathbb{Z}[\delta]$ for the parameter δ, and the Temperley-Lieb algebra $TL_n = \mathrm{End}_{\mathcal{TL}}(n, n)$. Let $I_{<n}$ be the ideal in TL_n spanned by all morphisms which factor through the object k for $k < n$. Given a commutative ring R, and a homomorphism $\mathbb{Z}[\delta] \to R$ (that is, a choice of $\delta \in R$), we call an idempotent $J_n \in TL_n \otimes_{\mathbb{Z}[\delta]} R$ a *Jones-Wenzl projector* if $J_n I_{<n} = I_{<n} J_n = 0$ and $1 - J_n \in I_{<n}$.

Proposition A.1. *For any $\delta \in R$, there is at most one Jones-Wenzl projector.*

Proof. Assume that J_n and J'_n are two JW projectors. Since $J_n - J'_n \in I_{<n}$ one has $(J_n - J'_n)J_n = (J_n - J'_n)J'_n = 0$. Therefore,

$$J_n = J_n^2 = J_n J'_n + J_n(J_n - J'_n) = J_n J'_n = J_n J'_n + J'_n(J'_n - J_n) = (J'_n)^2 = J'_n. \qquad \square$$

Thus we can speak of "the" Jones-Wenzl projector. The question we will wish to consider is when this projector exists and when it does not. For example, in TL_2, one can easily check that the JW projector exists if and only if δ is a unit. For higher ranks, the existence question is more subtle, but still easy to resolve. As noted earlier, we can define unique polynomials in δ that are sent to the quantum integers $[k]$ or quantum binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ under the specialization $\delta \mapsto q + q^{-1}$.

Theorem A.2. *The Jones-Wenzl projector exists over R if and only if the quantum binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is invertible in R for all $k < n$.*

In order to establish this theorem, we'll have to give some preliminary results; most of the techniques will be familiar to experts, but we will write out detailed proofs to make it clear that they work over an arbitrary ring.

Let
$$R' = R[q, q^{-1}]/(\delta - q - q^{-1}).$$
Note that R injects in R' as the fixed points of the bar involution sending $q \mapsto q^{-1}$ and fixing R. The Temperley-Lieb algebra $TL_n \otimes_{\mathbb{Z}[\delta]} R'$ has an induced bar involution preserving the diagram basis whose fixed points are $TL_n \otimes_{\mathbb{Z}[\delta]} R$. The uniqueness of J_n guarantees that if it exists over R' then it is preserved by the bar involution, and thus exists over R. Also, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is invertible in R if and only if it is invertible in R'.

The combinatorics of Jones-Wenzl projectors will be easier to understand if we realize them as endomorphisms of a representation of the quantum group. Let $U = U_q(\mathfrak{sl}_2)$ be the algebra generated over $A = \mathbb{Z}[q, q^{-1}]$ by $E^{(i)}, F^{(i)}, K$ under the usual relations. We set $\delta = q + q^{-1}$. Note that a homomorphism $\mathbb{Z}[\delta] \to R$ induces a unique map $A \to R'$ sending $q \to q$. Consider the standard representation on $V = A^2$ via the matrices
$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}.$$

The algebra U is a Hopf algebra, with the coproduct
$$\Delta(E) = E \otimes K + 1 \otimes E \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \tag{A.1}$$
there is an induced U-module structure on $M \otimes_A N$ for any U-modules M, N, and similarly for the base change to any A-algebra. In particular, we can also consider the tensor powers $V^{\otimes n}$ (tensor product over A).

Let $W(k)$ be the Weyl module over A, that is, the free A module spanned by a highest weight vector v_k of weight k, and its images under divided powers $F^{(i)}v_k$, for $i \leq k$, with $F^{(k+1)}v_k = 0$. The algebra U acts via
$$E^{(j)}F^{(i)}v_k = \begin{bmatrix} k - j + i \\ j \end{bmatrix} F^{(i-j)}v_k \qquad F^{(j)}F^{(i)}v_k = \begin{bmatrix} j + i \\ j \end{bmatrix} F^{(i+j)}v_k.$$
There is a standard duality on $U \otimes_A R'$ modules which are free of finite rank over R' by considering the action on $M^* := \mathrm{Hom}_A(M, R')$ induced by the antipode. Throughout, we will only consider finitely generated R'-modules, so we only consider free modules of finite rank. Thus, if M and N are $U \otimes_A R'$-modules whose underlying R' structure is free, we have isomorphisms compatible with the $U \otimes_A R'$-module structure
$$\mathrm{Hom}_A(M, N) \cong M^* \otimes N \qquad \mathrm{Hom}_U(M, N) \cong (M^* \otimes N)^U.$$
We let $W^*(k)$ be the dual Weyl module; this is spanned by the dual basis $\{w_{k-2p}\}$ for $0 \leq p \leq k$ to the basis $\{F^{(p)}v_k\}$.

Definition A.3. We say that a $U \otimes_A R'$-module M is *Weyl filtered* if it possesses a filtration $L_0 = 0 \subset L_1 \subset L_2 \subset \cdots \subset L_r = M$ with $L_j/L_{j-1} \cong W(p_j) \otimes_A R'$. We say that M is *dual Weyl filtered* if it possesses such a filtration with $L_j/L_{j-1} \cong W^*(q_j) \otimes_A R'$.

Note that the property of having a (dual) Weyl filtration is preserved under base change by the freeness of $W(p_j)$ over A.

Lemma A.4. *The Weyl module $W(k)$ has the universal property that $\mathrm{Hom}_U(W(k), M)$ is canonically isomorphic to the set of vectors in the module M of weight k killed by $E^{(m)}$ for $m > 0$ and $F^{(p)}$ for $p > k$. Dually, the module $W^*(k)$ also has a universal property in the category of U-modules with underlying A-module free; $\mathrm{Hom}_U(M, W(k)^*)$ is canonically isomorphic to space of A-homomorphisms $M \to A$ of weight k killed by E^m for $m > 0$ and $F^{(p)}$ for $p > k$.*

Proof. By duality, it suffices to prove the property of a Weyl module. This follows immediately from the presentation: the action of U on v_k presents $W(k)$ as the quotient of U by the left ideal I generated by $E^{(m)}$ for $m > 0$ and $F^{(p)}$ for $p > k$. Homomorphisms $U/I \to M$ are in canonical bijection with $\{m \in M | Im = 0\}$, which is the desired set of vectors. □

This allows us to show that:

Lemma A.5. *For all k, k', we have*

$$\mathrm{Hom}_U(W(k) \otimes_A R', W^*(k') \otimes_A R') = \begin{cases} R' & k = k' \\ 0 & k \neq k' \end{cases}.$$

$$\mathrm{Ext}^1_U(W(k) \otimes_A R', W^*(k') \otimes_A R') = 0.$$

Proof. The description of homomorphisms follows from Lemma A.4 applied to $W(k)$ if $k \geq k'$ and to $W^*(k')$ if $k' \geq k$. If we have an extension

$$0 \to W^*(k') \otimes_A R' \to E \to W(k) \otimes_A R' \to 0$$

then Lemma A.4 shows that if $k' \geq k$, then we can build a splitting $E \to W^*(k') \otimes_A R'$, and if $k \geq k'$, we can get a splitting $W(k) \otimes_A R' \to E$. □

Now, assume M is Weyl filtered, with $W(k)$ having multiplicity m_k and N is dual Weyl filtered with $W(k)^*$ having multiplicity n_k. Since we are assuming M and N are finitely generated over R', this implies that $m_N = n_N = 0$ for $N \gg 0$.

Corollary A.6. *With M, N as above, $\mathrm{Hom}_U(M \otimes_A R', N \otimes_A R')$ is a free R'-module of rank $\sum_{k=0}^{\infty} m_k n_k$.*

We call a U-module *tilting* if it has a filtration whose consituents are Weyl modules and another whose constituents are dual Weyl modules.

Lemma A.7. *The module $V^{\otimes n}$ is tilting.*

Proof. First, note that $V^{\otimes n}$ is self-dual, since V is as well. Thus, it suffices to show that $V^{\otimes n}$ has a Weyl filtration, or more generally that if N has a Weyl filtration, $N \otimes V$ does as well. To see this, it suffices to show that for any Weyl module $W(k)$, we have a short exact sequence

$$0 \to W(k+1) \to W(k) \otimes V \to W(k-1) \to 0.$$

Since $V = W(1)$, we denote its highest weight vector by v_1. The inclusion $W(k+1) \hookrightarrow W(k) \otimes V$ sends

$$F^{(i)}v_{k+1} \mapsto F^{(i)}(v_k \otimes v_1) = F^{(i)}v_k \otimes v_1 + q^{-k+i-1}F^{(i-1)}v_k \otimes Fv_1.$$

The quotient module Q has a basis as a free A-module given by the images of the vectors $F^{(p)}v_k \otimes Fv_1$ for $p = 0, \ldots, k-1$. Since Q is generated by the highest weight vector $v_k \otimes Fv_1$, there is a map $W(k-1) \to Q$, which is an surjective map between free A modules of rank k and thus an isomorphism. The tilting property follows. $\square$

We can define a functor $\rho\colon \mathcal{TL} \to U\text{-mod}$ which sends $n \mapsto V^{\otimes n}$. This functor will be monoidal, so we need only specify the image of the cup $\iota\colon A \to V^{\otimes 2}$ and the cap $\epsilon\colon V^{\otimes 2} \to A$. These are given by the unique homomorphisms such that

$$\iota(1) = q^{-1}Fv_1 \otimes v_1 - v_1 \otimes Fv_1 \qquad \epsilon(-Fv_1 \otimes v_1) = \epsilon(q^{-1}v_1 \otimes Fv_1) = 1. \tag{A.2}$$

The existence of this functor is a standard result due (in different form) to Temperley and Lieb's original paper [23]. Obviously, we can extend scalars to obtain a functor $\rho \otimes_A R'\colon \mathcal{TL} \otimes_{\mathbb{Z}[\delta]} R' \to U \otimes_A R'\text{-mod}$.

Lemma A.8. *The functor $\rho \otimes_A R'$ is fully faithful.*

Proof. Using duality, it suffices to prove that the induced map from $\mathrm{Hom}_{\mathcal{TL}}(0, n) \to \mathrm{Hom}_U(R', V^{\otimes n} \otimes_A R')$ is an isomorphism. By localization and Nakayama's lemma, it suffices to prove this isomorphism after base change to any field $\mathbb{K}$. Note that by Corollary A.6, both of these modules are free of dimension given by the nth Catalan number. Thus, it suffices to show the map is injective, that is, that the vector v_C attached to different crossingless matchings C by (A.2) are linearly independent. For a fixed C, we let $\epsilon_1, \cdots, \epsilon_n$ be the sequence of n elements of $\{1, 0\}$ where we put a 1 over the left end of a cup and 0 over the right end, (so we have $(1, 1, 0, 0)$ for two nested cups, and $(1, 0, 1, 0)$ for unnested). In this case, we have $v_C = q^{-n}F^{\epsilon_1}v_1 \otimes \cdots \otimes F^{\epsilon_n}v_1 + \cdots$ where the other terms correspond to words in $\{1, 0\}$ which are smaller in lexicographic order. This shows that no multiple av_C be written in terms of $v_{C'}$ for $C' < C$ in lexicographic order. Thus, any relation between crossingless matchings must be trivial, since we arrive at a contradiction when we move av_C for C lexicographically maximal to the other side of the equation. $\square$

Given a U module (or a $U \otimes_A R'$ module) M, let M_p denote its p-th weight space. Let $M[<n]$ denote the maximal submodule whose weight spaces for $p \geq n$ are zero.

Proposition A.9. *Assume that M is a $U \otimes_A R'$ module with a dual Weyl filtration. Then the sum of the images of all maps $V^{\otimes k} \otimes_A R' \to M$ for $k < n$ is precisely $M[<n]$. Furthermore, if N is Weyl filtered, then any map $N \to M[<n]$ is a sum of maps factoring through $V^{\otimes k} \otimes_A R'$ for $k < n$.*

Proof. Assume M' is a submodule of M with all weight spaces for $p \geq k$ trivial. We wish to show that M' is in the sum of the images of maps $V^{\otimes k} \otimes_A R' \to M$ for $k < n$. Let M'' be a submodule with a dual Weyl filtration $L_0 = 0 \subset L_1 \subset L_2 \subset \cdots \subset L_r =$

M'' satisfying $M' \subset M'' \subset M$. We choose M'' of minimal length r, and prove the result by induction on r.

Let p be the maximal weight of M''. By the universal property, we have a map $q\colon M'' \to M''_p \otimes_A W(p)^*$. Let r' be minimal such that $M''_p \subset L_{r'+1}$, that is, maximal such that $p_{r+1} = p$. Thus, $M''_p \cap L_{r'}$ is a free summand of M''_p of corank 1. Consider the map $M'' \to (M''_p / M''_p \cap L_{r'}) \otimes_A W(p)^*$ and let $K_{r'}$ be its kernel. We have that $L_m \subset K_{r'}$ if $m \leq r'$, and we have an induced isomorphism $L_{r'+1}/L_{r'} \cong (M''_p/M''_p \cap L_{r'}) \otimes_A W(p)^*$. Thus, we have that

$$(L_j \cap K_{r'})/(L_{j-1} \cap K_{r'}) \cong \begin{cases} L_j/L_{j-1} & j \neq r'+1 \\ 0 & j = r'+1. \end{cases}$$

Thus K has a dual Weyl filtration. If $p \geq n$, then the induced map $M' \to M''_p \otimes_A W(p)^*$ must be 0, so $M' \subset K$. This contradicts the minimality of M''.

Thus, we must have that $p < n$. In this case, we also have a surjective map $\pi_{M''}\colon M''_p \otimes_A V^{\otimes p} \to M''_p \otimes_A W(p)^*$. As argued above $\ker q$ is dual Weyl filtered, so $\mathrm{Ext}^1(M''_p \otimes_A V^{\otimes p}, \ker q) = 0$. Thus, we can lift $\pi_{M''}$ to a map $M''_p \otimes_A V^{\otimes p} \to M''$. The image of this map together with $\ker q$ spans M''. By induction, $\ker q$ is spanned by the images of maps from $V^{\otimes k} \otimes_A R'$ with $k < p < n$. Thus, we have proved these maps have the correct sum.

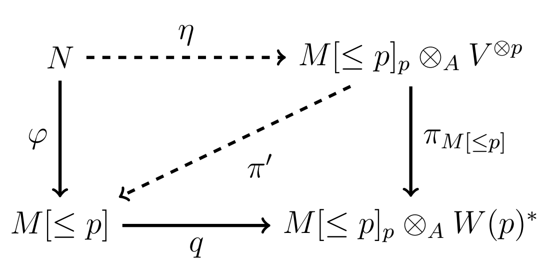
Now let $\varphi\colon N \to M[< n]$ be a map from a Weyl filtered module; let p be the maximal weight space in N on which this map is non-zero. We will prove the result by induction on p. The image of our map must thus lie in $M[\leq p]$. Consider the projection $q\colon M[\leq p] \to M[\leq p]_p \otimes_A W(p)^*$. As above, we also have a map

$$\pi_{M[\leq p]}\colon M[\leq p]_p \otimes_A V^{\otimes p} \to M[\leq p]_p \otimes_A W(p)^*$$

and the argument above shows that we can define a map $\pi'\colon M[\leq p]_p \otimes_A V^{\otimes p} \to M[\leq p]$ such that $\pi_{M[\leq p]} = q \circ \pi'$ as shown in (A.3) below.

The composition $q \circ \varphi$ is a map from a Weyl filtered module to $M[\leq p]_p \otimes_A W(p)^*$. Note that the kernel K of the map $\pi_{M[\leq p]}$ is also dual Weyl filtered so $\mathrm{Ext}^1(N, K) = 0$. Thus, the map $q \circ \varphi$ can be lifted to a map $\eta\colon N \to M[\leq p]_p \otimes_A V^{\otimes p}$ which thus satisfies that $q \circ \varphi = \pi_{M[\leq p]} \circ \eta$.

(A.3)



Thus, the map $\varphi' = \varphi - \pi_{M[\leq p]} \circ \eta$ lands in $M[< p]$. By induction, φ' is a sum of maps that factor through $V^{\otimes k}$ for $k < p$, so this completes the proof. $\square$

Proof of Theorem A.2. If the Jones-Wenzl projector J_n exists, then we can consider its action on the module $M = V^{\otimes n} \otimes_A R'$. By definition $1 - J_n$ is the unique idempotent in $I_{<n}$ such that $(1 - J_n)I_{<n} = I_{<n}$. The fact that $1 - J_n \in I_{<n}$ shows its image must lie in $M[< n]$, while the fact that $(1 - J_n)I_{<n} = I_{<}n$ together with the previous proposition show that it must act on $M[< n]$ by an isomorphism. That is, $1 - J_n$ must be the projection to this $M[< n]$ with some complement $X = \mathrm{im}(J_n)$. Since there is a short exact sequence

$$0 \to M[< n] \to M \to W^*(n) \otimes_A R' \to 0,$$

we have that $X \cong W^*(n) \otimes_A R'$. On the other hand, since J_n acts by the identity on M_n, the induced map $W(n) \otimes_A R' \to X$ must be an isomorphism as well. Thus, we have that the natural map $W(n) \otimes_A R' \to M \to W^*(n) \otimes_A R'$ must be an isomorphism. Under this map $F^{(k)}v_k$ is sent to the map

$$F^{(k)}w_n = w_n \circ (-1)^k q^{k(k-1)/2} K^{-k} E^{(k)} \colon W(n) \to A,$$

since $S(F^{(k)}) = -q^{k(k-1)/2} K^{-k} E^{(k)}$. Calculating, this is given by

$$(-1)^k q^{k(k-1)/2 - nk} \begin{bmatrix} n \\ k \end{bmatrix} w_{n-2k}.$$

This spans the $n - 2k$ weight space (and thus the map is an isomorphism) if and only if $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is a unit in R'.

On the other hand, assume that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is always a unit. In this case, the induced map $W(n) \otimes_A R' \to M \to W^*(n) \otimes_A R'$ is an isomorphism by the same calculation. Thus, the image of $W(n) \otimes_A R'$ is a complementary submodule to $M[< n]$ inside M. The projection to $M[< n]$ must be in the ideal $I_{<n}$, so the complementary projection to $W(n) \otimes_A R'$ must be the image of a Jones-Wenzl projector under $\rho \otimes_A R'$, so J_n must exist. □

References

[1] Alexander Beilinson and Joseph Bernstein. Localisation de g-modules. *CR Acad. Sci. Paris*, 292(1):15–18, 1981.

[2] Nicolas Bourbaki. *Elements de mathematique, fasc. 34: groupes et algèbres de Lie, chap. 4, 5 et 6*. Hermann, 1968.

[3] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Inventiones mathematicae*, 64(3):387–410, 1981.

[4] Matthew Dyer. On some generalisations of the Kazhdan-Lusztig polynomials for "universal" Coxeter systems. *Journal of Algebra*, 116(2):353–371, 1988.

[5] Ben Elias. Dihedral cathedral. Preprint. arXiv:1308.6611.

[6] Ben Elias. Quantum algebraic geometric satake. Preprint. arXiv:1403.5570.

[7] Ben Elias and Geordie Williamson. The Hodge theory of Soergel bimodules. Preprint. arXiv:1212.0791.

[8] Ben Elias and Geordie Williamson. Soergel calculus. Preprint. arXiv:1309.0865.

[9] Peter Fiebig. The combinatorics of Coxeter categories. *Transactions of the American Mathematical Society*, 360(8):4211–4233, 2008.

[10] Frederick M Goodman and Hans Wenzl. Ideals in the Temperley Lieb category. 2002. arXiv:math/0206301.

[11] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[12] Daniel Juteau, Carl Mautner, and Geordie Williamson. Perverse sheaves and modular representation theory. In *Geometric methods in representation theory II*, volume 25 of *Séminaires et Congrès*, pages 313–350. Soc. Math. France, Paris, 2010.

[13] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. volume 53, pages 165–184. 1979.

[14] Aaron D. Lauda. A categorification of quantum $\mathrm{sl}(2)$. *Adv. Math.*, 225(6):3327–3424, 2010.

[15] Aaron D. Lauda. An introduction to diagrammatic algebra and categorified quantum sl_2. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 7(2):165–270, 2012.

[16] Helmut Lenzing.

[17] Nicolas Libedinsky. Light leaves and Lusztig's conjecture. Preprint, arXiv:1304.1448.

[18] Nicolas Libedinsky. Sur la catégorie des bimodules de Soergel. *J. Algebra*, 320(7):2675–2694, 2008.

[19] Scott Morrison. A formula for the Jones-Wenzl projections. preprint, available at http://tqft.net/math/JonesWenzlProjections.pdf.

[20] Wolfgang Soergel. The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math.*, 429:49–74, 1992.

[21] Wolfgang Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152(1-3):311–335, 2000. Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998).

[22] Wolfgang Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.

[23] H. N. V. Temperley and E. H. Lieb. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A*, 322(1549):251–280, 1971.

[24] Hans Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

[25] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, 219(4):539–565, 1995.

[26] Bruce W. Westbury. Invariant tensors and cellular categories. *J. Algebra*, 321(11):3563–3567, 2009.

[27] Geordie Williamson. Schubert calculus and torsion. Preprint, arXiv:1309.5055.
[28] Geordie Williamson. Singular Soergel bimodules. *Int. Math. Res. Not. IMRN*, (20):4555–4632, 2011.

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