THE ANTI-SPHERICAL CATEGORY

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ABSTRACT. We study a diagrammatic categorification (the "anti-spherical category") of the anti-spherical module for any Coxeter group. We deduce that Deodhar's (sign) parabolic Kazhdan-Lusztig polynomials have non-negative coefficients, and that a monotonicity conjecture of Brenti's holds. The main technical observation is a localisation procedure for the anti-spherical category, from which we construct a "light leaves" basis of morphisms. Our techniques may be used to calculate many new elements of the p-canonical basis in the anti-spherical module. The results use generators and relations for Soergel bimodules ("Soergel calculus") in a crucial way.

1. INTRODUCTION

1.1. Kazhdan-Lusztig polynomials are remarkable polynomials associated to pairs of elements in a Coxeter group. They describe the base change matrix between the standard and Kazhdan-Lusztig basis of the Hecke algebra. Since their discovery by Kazhdan and Lusztig in 1979, these polynomials have found applications throughout representation theory.

A fascinating aspect of the theory is that these polynomials are elementary to define and compute, however they also have deep properties that are far from obvious from their definition. For example, it was conjectured by Kazhdan and Lusztig in [KL79] that these polynomials have non-negative coefficients. This conjecture was established soon after by Kazhdan and Lusztig [KL80] if the underlying Coxeter group is a finite or affine Weyl group. Recently, Kazhdan and Lusztig's conjecture was established in complete generality by Elias and the second author via Soergel bimodule techniques [EW14].

In 1987 Deodhar introduced parabolic Kazhdan-Lusztig polynomials [Deo87]. These polynomials are defined starting from the choice of a Coxeter group, a standard parabolic subgroup and a sign. They describe the base change matrix between the standard and Kazhdan-Lusztig basis of the spherical or anti-spherical (depending on the sign) module for the Hecke algebra. Kazhdan-Lusztig polynomials agree with parabolic Kazhdan-Lusztig polynomials for the choice of the trivial parabolic subgroup. Parabolic Kazhdan-Lusztig polynomials are also known to have deep representation theoretic and geometric significance. One of the two main theorems of this paper is the following:

Theorem 1.1. *Parabolic Kazhdan-Lusztig polynomials associated to the sign representation have non-negative coefficients, for any Coxeter system and any choice of standard parabolic subgroup.*

Two remarks on this theorem:

(1) The analogue of Kazhdan and Lusztig's theorem identifying Kazhdan-Lusztig polynomials with the Poincaré polynomials of the stalks of intersection cohomology complexes on the flag variety was given in a beautiful paper by Kashiwara and Tanisaki [KT02] in 2002 (fifteen years both after the introduction of parabolic Kazhdan-Lusztig polynomials, and before the current paper!). Thus the above theorem was already known for any Coxeter group which arises as the Weyl group of a symmetrisable Kac-Moody Lie algebra. This is the case if and only if the order of any two simple reflections belongs to the set $\{2, 3, 4, 6, \infty\}$.
(2) The methods of this paper are easily adapted to deduce a similar theorem for parabolic Kazhdan-Lusztig polynomials associated to the trivial representation. For various reasons we regard the case of the trivial representation as being easier. We hope to discuss this case elsewhere.

1.2. The proof that Kazhdan-Lusztig polynomials have non-negative coefficients in [EW14] relies on a detailed study of a categorification of the Hecke algebra via certain bimodules constructed by Soergel [Soe90, Soe07], which have come to be known as Soergel bimodules. The essential point ("Soergel's conjecture") is that the Kazhdan-Lusztig basis arises as the classes in the Grothendieck group of indecomposable Soergel bimodules. Thus Soergel bimodules provide a meaning (as graded dimensions of certain Hom spaces) for Kazhdan-Lusztig polynomials associated to any Coxeter system.

More recently, Elias and the second author described the monoidal category of Soergel bimodules by generators and relations [EW16]. The result is a diagrammatically defined additive graded monoidal category which is equivalent to the monoidal category of Soergel bimodules. In this paper we work almost exclusively with this category, which we denote $\mathcal{H}$ and call the Hecke category.

It is natural to try to understand parabolic Kazhdan-Lusztig polynomials by categorifying the modules in which they live. This is precisely what we do in this paper for the anti-spherical module.

1.3. Let (W, S) be a Coxeter system, and let H be its Hecke algebra over $\mathbb{Z}[v^{\pm 1}]$. Let h_x denote its standard basis and b_x its canonical (or Kazhdan-Lusztig) basis. Fix a subset $I \subset S$ and let ${}^I W$ denote the set of minimal coset representatives for $W_I \setminus W$. Let N denote the anti-spherical (right) H-module

$$N := \mathrm{sgn_v} \otimes_{H_I} H,$$

where $\mathrm{sgn_v}$ denotes the quantized sign representation of H_I, the standard parabolic subalgebra of H determined by I. Let n_x denote the standard basis of N and d_x its Kazhdan-Lusztig basis.

Recall the Hecke category $\mathcal{H}$ from above. For any $w \in W$ there exists an indecomposable self-dual object $B_w \in \mathcal{H}$ parametrized by w. Any indecomposable self-dual object in $\mathcal{H}$ is isomorphic to B_w for some $w \in W$. We have a canonical isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$H \xrightarrow{\sim} [\mathcal{H}]$$

defined on generators by $b_s \mapsto [B_s]$, for all $s \in S$. Here we have employed the following notation: Given an additive graded (with shift functor $M \mapsto M(1)$) category $\mathcal{M}$, let $[\mathcal{M}]$ denote its split Grothendieck group, which we view as a $\mathbb{Z}[v^{\pm 1}]$-module via $v[M] := [M(1)]$. Note that $[\mathcal{H}]$ is an algebra because $\mathcal{H}$ is a monoidal category.

Now inside $\mathcal{H}$ consider $\mathcal{I}$ the additive category consisting of all direct sums of shifts of B_x, for $x \notin {}^I W$. It turns out that $\mathcal{I}$ is a right tensor ideal of $\mathcal{H}$ (i.e. if $X \in \mathcal{I}$ and $B \in \mathcal{H}$ then $XB \in \mathcal{I}$). In particular, if we consider the quotient[1] of additive categories

$$\mathcal{N} := \mathcal{H}/\mathcal{I}$$

then this is a right module category over $\mathcal{H}$. We call $\mathcal{N}$ the *anti-spherical category* (associated to the subset $I \subset S$). The following theorem justifies the name:

Theorem 1.2. *There is a canonical isomorphism $N \xrightarrow{\sim} [\mathcal{N}]$ of $\mathbb{Z}[v^{\pm 1}]$-modules. This is an isomorphism of right H-modules via the identification $H = [\mathcal{H}]$. Under this isomorphism, the indecomposable self-dual objects in $\mathcal{N}$ correspond to the Kazhdan-Lusztig basis in N.*

We also prove a theorem giving a ("light leaves") basis for the morphisms between certain additive generators of $\mathcal{N}$ (see Theorem 7.3). From this the positivity of the corresponding parabolic Kazhdan-Lusztig polynomials (Theorem 1.1) is an easy consequence. We also deduce (see Corollary 8.4) from these results a proof of a conjecture of Brenti [Mon14] on the monotonicity of parabolic Kazhdan-Lusztig polynomials associated to increasing subsets $I \subseteq J \subseteq S$.

1.4. We were also motivated in our study of the anti-spherical category by representation theory. If W is the Weyl group of a complex semi-simple Lie algebra, the anti-spherical category can be used to give a graded deformation of parabolic category $\mathcal{O}$ (the subset $I \subset S$ is determined by the parabolic subgroup appearing in the definition of parabolic category $\mathcal{O}$). This fact does not seem to be available explicitly in the literature, however the papers [Str05] and [KMS08] contain results which are quite close.

The anti-spherical category is also important in modular representation theory. Recent results and conjectures of Riche and the second author connect the anti-spherical module for the affine Weyl group to representations of algebraic groups [RW15]. There the authors conjecture (and prove for GL_n) that a certain mod p version of the anti-spherical category provides a graded version of the category of tilting modules for the Langlands dual group. Thus (assuming the conjecture) the anti-spherical category sees all of the (extremely subtle) representation theory of connected reductive algebraic groups. (These conjectures were heavily motivated by earlier work of Soergel [Soe97] and Arkhipov-Bezrukavnikov [AB09].) More recently, Elias and Losev [EL] have explained that one can use (singular) Soergel bimodules to construct the categories of polynomial representations of GL_n together with the action of certain natural endofunctors, in a purely combinatorial way. Their work provides further evidence for the importance of the anti-spherical category in modular representation theory.

In [RW15] (the obvious analogue of) Theorem 1.2 is proved for the anti-spherical module of an affine Weyl group. (The parabolic subgroup is taken to be the finite Weyl group.) The proofs there rely on geometry or representation theory in a crucial way. One of the main motivations for the current work was to give purely algebraic proofs of these basic statements, which work for any Coxeter system. The proofs of the current paper involve quite different technology than those of [RW15]

[1]By quotient we mean the following: the objects of $\mathcal{N}$ are the same as those of $\mathcal{H}$; a morphism is zero in $\mathcal{N}$ if and only if it factors through an object of $\mathcal{I}$.

and are simpler and more general. A key to our approach is the infinite twist (see § 5), which first emerged in work on link homology [DGR06, Roz14, Cau15, Cau16]. This appearance of the infinite twist is an appealing aspect of the current work.

1.5. Another consequence of the conjectures of [RW15] is a character formula for simple modules and indecomposable tilting modules for reductive algebraic groups in characteristic p in terms of the p-canonical basis of the anti-spherical module. This conjecture has recently been proved by Achar, Makisumi, Riche and the second author [AMRWa, AMRWb]. The recent paper [EL] of Elias and Losev has related (and in certain cases stronger) results for GL_n.

The upshot is that the p-canonical basis in the anti-spherical module contains the answers to several deep mysteries in the representation theory of algebraic groups. However it is still not easy to compute. The second main theorem of this paper (Theorem 3.3) heuristically says that the localisation of the anti-spherical category is "as simple as possible". This lead to a more effective means of calculating the p-canonical basis. The basic idea is that via localisation one can reduce calculations of the p-canonical basis in the anti-spherical module (which can be performed via diagrammatics, as explained in [JW15]) to certain linear algebra problems over a polynomial ring in one variable (the ring denoted R_I in § 3.7). This algorithm has been further developed and implemented by the second author [Wil] to provide a powerful new means to calculate decomposition numbers for symmetric groups.

1.6. We conclude this introduction with a remark on positive characteristic. In the body of this paper we work over a field of characteristic zero. This is because our results rely crucially on the so-called parabolic property of root systems (see (2.3)), which often fails for reflection representations of Coxeter groups in positive characteristic. The parabolic property ensures that our second main theorem (Theorem 3.3) holds. It is an interesting question as to what happens if one localises in settings in which the parabolic property fails (as is the case for the important example of the natural representation of affine Weyl groups in characteristic p). We do not address this question in this paper. Finally, let us remark that one can still apply the techniques of this paper to settings in positive characteristic by using the p-adic integers in place of a field of characteristic p. This is one of the basic ideas in [Wil].

1.7. **Acknowledgements:** This paper owes an intellectual debt to ideas of R. Bezrukavnikov and S. Riche. We would like to thank them both for useful discussions. We would also like to thank B. Leclerc for pointing out [KT02] and M. Hogancamp for interesting discussions around the infinite twist. Finally we would like to thank T. Jensen and P. Sentinelli for detailed comments.

2. Parabolic Kazhdan-Lusztig polynomials

2.1. **The Hecke algebra.** We follow the notation of [Soe97]. Let (W,S) be a Coxeter system and $(m_{sr})_{s,r\in S}$ its Coxeter matrix. Let $l : W \to \mathbb{N}$ be the corresponding length function and $\leq$ the Bruhat order on W. Let $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$ be the ring of Laurent polynomials with integer coefficients in one variable v.

The *Hecke algebra* $H = H(W,S)$ of a Coxeter system (W,S) is the associative algebra over $\mathcal{L}$ with generators $\{h_s\}_{s\in S}$, quadratic relations $(h_s+v)(h_s-v^{-1}) = 0$ for all $s \in S$, and braid relations $h_s h_r h_s \cdots = h_r h_s h_r \cdots$ with m_{sr} elements on each side for every couple $s, r \in S$.

Consider $x \in W$. To a reduced expression $sr \cdots t$ of x one can associate the element $h_s h_r \cdots h_t \in H$. It was proved by H. Matsumoto that this element is independent of the choice of reduced expression of x and we call it h_x. N. Iwahori proved that

$$H = \bigoplus_{x \in W} \mathcal{L} h_x$$

and $h_x h_y = h_{xy}$ if $l(x) + l(y) = l(xy)$.

Let us define the element $b_s = h_s + v$. The right regular action of H is given by the formula:

$$h_x b_s = \begin{cases} h_{xs} + v h_x & \text{if } x < xs; \\ h_{xs} + v^{-1} h_x & \text{if } x > xs. \end{cases} \tag{2.1}$$

2.2. **Parabolic subgroups.** Consider $I \subset S$ an arbitrary subset and W_I its corresponding Coxeter group. We say that W_I is the *parabolic subgroup* corresponding to I. We say that a sequence $\underline{w}$ of elements in S is an *I-sequence* if it starts with some element $s \in I$.

We denote by ${}^I W \subseteq W$ the set of minimal coset representatives in $W_I \backslash W$. The following two descriptions of this set will be useful for us:

$$ {}^I W = \{ w \in W \mid sw > w \text{ for all } s \in I \}; \tag{2.2}$$

$$ {}^I W = \{ w \in W \mid \text{ no reduced expression of } w \text{ is an } I\text{-sequence} \}. \tag{2.3}$$

Example 2.1. *Let W be the symmetric group $W = S_8$ with simple reflections $s_1, s_2, \ldots, s_7$. For simplicity we will just denote s_k by k, so by 343 we mean the element $s_3 s_4 s_3 \in W$. Let us define the set*

$$\underrightarrow{54321} := \{\emptyset, 5, 54, 543, 5432, 54321\} \subseteq W.$$

We define in the same way the set $\underrightarrow{k \ldots 321}$ for any natural number k. The order of this set is $k+1$.

Say that $I = \{1, 2, 3\}$. Then W_I and ${}^I W$ are the following products of sets

$$W_I = \underrightarrow{1}\ \underrightarrow{21}\ \underrightarrow{321} \quad \textit{(it has order } 2 \cdot 3 \cdot 4 = 24\textit{) and}$$

$${}^I W = \underrightarrow{4321}\ \underrightarrow{54321}\ \underrightarrow{654321}\ \underrightarrow{7654321} \ \textit{(it has order } 5 \cdot 6 \cdot 7 \cdot 8 = 1680\textit{)}.$$

For example, $1\,21\,32 \in W_I$ *and* $43\,543\,6\,765432 \in {}^I W$.

We see in this example (if one recalls the normal form of an element in the symmetric group) that multiplication defines an isomorphism

$$W \cong W_I \times {}^I W \tag{2.4}$$

Dehodar [Deo87] proved that this is true for any Coxeter system and any parabolic subgroup. Another important property of minimal coset representatives is what we will call

2.3. **Parabolic Property.** Let $\bigoplus_{r\in S}\mathbb{R}\alpha_r^\vee$ be the geometric representation of W (see Section 3.1). Let $\Delta_I := \{\alpha_r\}_{r\in I}$ and let Φ_I be the root system spanned by Δ_I. If $x \in {}^IW$ and $s\in S$, then

$$xs \notin {}^IW \Longleftrightarrow x(\alpha_s)\in\Delta_I \Longleftrightarrow x(\alpha_s)\in\Phi_I.$$

Proof.

- We first prove that $xs\notin {}^IW \Rightarrow x(\alpha_s)\in\Delta_I$. If $x\in {}^IW$ and $xs\notin {}^IW$ then $xs = rx$ for some $r\in I$. This comes from the more general (and beautiful) fact that if x is any element of W and $s, r\in S$, the two inequalities $rx > x$ and $rxs < xs$ imply that $rxs = x$. Hence $x(\alpha_s) = rxs(\alpha_s)$. Thus we obtain the equality $r(x(\alpha_s)) = -x(\alpha_s)$, that implies that $x(\alpha_s) = \alpha_r$.
- The proof that $x(\alpha_s)\in\Delta_I \Rightarrow x(\alpha_s)\in\Phi_I$ is trivial.
- Finally we prove that $x(\alpha_s)\in\Phi_I\Rightarrow xs\notin {}^IW$. As $x(\alpha_s)\in\Phi_I$, we know that $xsx^{-1} = t\in W_I$ with t a reflection. Rewriting this equation we have $xs = tx$. But Isomorphism 2.4 implies that $l(tx) = l(t) + l(x)$. This says that $l(t) = 1$, thus t is a simple reflection in I. □

2.4. **Spherical and anti-spherical modules.** Consider $I\subset S$ and the Hecke algebra $H_I := H(W_I, I)$. By the relations defining the Hecke algebra, if we fix $u\in\{-v, v^{-1}\}$, we can define a surjection of $\mathcal{L}$-algebras

$$\varphi_u : H_I \twoheadrightarrow \mathcal{L}$$

by sending $h_s\mapsto u$ for all $s\in I$. Thus $\mathcal{L}$ becomes an H_I-bimodule which we denote by $\mathcal{L}(u)$. We can induce from it to produce the following right H-modules:

$$N = N(W,S,I) = \mathcal{L}(-v)\otimes_{H_I} H, \text{ the } \textit{anti-spherical module};$$

$$M = M(W,S,I) = \mathcal{L}(v^{-1})\otimes_{H_I} H, \text{ the } \textit{spherical module}.$$

If $n_x := 1\otimes h_x \in N$ and $m_x := 1\otimes h_x\in M$, then we have that

$$N = \bigoplus_{x\in {}^IW}\mathcal{L}n_x \quad\text{and}\quad M = \bigoplus_{x\in {}^IW}\mathcal{L}m_x.$$

We will not prove this result but we will explain why it is reasonable. The equality (2.2) tells us that if $x\notin {}^IW$, then there is $r\in I$ such that $rx < x$ then $n_x = -vn_{rx}$. In this way we see that the set $\{n_x\}_{x\in {}^IW}$ generates N over $\mathcal{L}$ (a similar result holds for M).

2.5. **Right action of the Hecke algebra.** The right action of H on the anti-spherical and on the spherical modules (compare with the regular action (2.1)) is given by the formulas

$$n_xb_s = \begin{cases} n_{xs} + vn_x & \text{if } x < xs \text{ and } xs\in {}^IW;\\ n_{xs} + v^{-1}n_x & \text{if } x > xs \text{ and } xs \in {}^IW;\\ 0 & \text{if } xs\notin {}^IW.\end{cases} \tag{2.5}$$

$$m_xb_s = \begin{cases} m_{xs} + vm_x & \text{if } x < xs \text{ and } xs\in {}^IW;\\ m_{xs} + v^{-1}m_x & \text{if } x > xs \text{ and } xs \in {}^IW;\\ (v+v^{-1}) & \text{if } xs\notin {}^IW.\end{cases} \tag{2.6}$$

Let us explain these formulas for the anti-spherical module. Similar arguments work in the spherical case. The first two equations of (2.5) are an easy consequence of (2.1). The third equation of (2.5) is a consequence of the following three facts:

(a) $\varphi_u(b_s) = 0$ for $s \in I$
(b) If $x \in {}^I W$ and $xs \notin {}^I W$ then $xs = rx$ for some $r \in I$.
(c) If $x \in {}^I W$ and $xs < x$ then $xs \in {}^I W$

Fact (a) is trivial. We have already seen fact (b) in § 2.3. Fact (c) is a direct consequence of fact (b).

2.6. **Kazhdan-Lusztig bases.** There is a unique ring homomorphism $h \mapsto \overline{h}$ on H such that $\overline{v} = v^{-1}$ and $\overline{h_x} = (h_{x^{-1}})^{-1}$. Recall that $b_s = h_s + v$. If $s \in I$ we have that $\varphi_{-v}(b_s) = 0$ and $\varphi_{v^{-1}}(b_s) = (v + v^{-1})$. In any case $\varphi_u(\overline{b_s}) = \overline{\varphi_u(b_s)}$ so, since the set $\{b_s\}_{s\in S}$ generates H_I as an $\mathcal{L}$-algebra, we have

$$\varphi_u(\overline{h_I}) = \overline{\varphi_u(h_I)} \text{ for any element } h_I \in H_I.$$

This means that we can induce the morphism $\overline{(-)}$ to a morphism of additive groups $\overline{(-)} : N \to N$ by $l \otimes h \mapsto \overline{l} \otimes \overline{h}$. In the same way we can induce a morphism of additive groups $\overline{(-)} : M \to M$. We will call an element *self-dual* if it is invariant under $\overline{(-)}$.

We can now state the central theorem of Kazhdan-Lusztig theory and its parabolic versions.

Theorem 2.2. (1) *(*[KL79]*) For every element $x \in W$ there is a unique self-dual element $b_x \in H$, such that $b_x \in h_x + \sum_{y\in W} v\mathbb{Z}[v]h_y$.*
(2) *(*[Deo87]*) For every element $x \in {}^I W$ there is a unique self-dual element $c_x \in M$, such that $c_x \in m_x + \sum_{y\in {}^I W} v\mathbb{Z}[v]m_y$.*
(3) *(*[Deo87]*) For every element $x \in {}^I W$ there is a unique self-dual element $d_x \in N$, such that $d_x \in n_x + \sum_{y\in {}^I W} v\mathbb{Z}[v]n_y$.*

We call the sets $\{b_x\}_{x\in W}$, $\{c_x\}_{x\in {}^I W}$ and $\{d_x\}_{x\in {}^I W}$ the Kazhdan-Lusztig bases of the corresponding H-modules. For each couple of elements $x, y \in W$ we define $h_{y,x} \in \mathcal{L}$ by the formula

$$b_x = \sum_y h_{y,x} h_y.$$

For each couple of elements $x, y \in {}^I W$ we define $m_{y,x} \in \mathcal{L}$ and $n_{y,x} \in \mathcal{L}$ by the formulae

$$c_x = \sum_y m_{y,x} m_y \quad \text{and} \quad d_x = \sum_y n_{y,x} n_y.$$

(If we need to specify the set I, we will write $m^I_{y,x}$ for $m_{y,x}$ and $n^I_{y,x}$ for $n_{y,x}$). The proof of Theorem 2.2 (as given by Soergel in [Soe97]) is short and easy. It constructs the Kazhdan-Lusztig basis inductively on the length of x.

The Kazhdan-Lusztig polynomials (as defined in [KL79]) are given by the formula $P_{y,x} = (v^{l(y)} - v^{l(x)})h_{y,x}$ and they are polynomials in $q := v^{-2}$. The same normalization gives Dehodar's parabolic polynomials. More precisely $(v^{l(y)} - v^{l(x)})m_{y,x}$ and $(v^{l(y)} - v^{l(x)})n_{y,x}$ are the polynomials $P^I_{y^{-1},x^{-1}}$ defined by Dehodar in [Deo87] in the cases $u = -1$ and $u = q$, respectively.

2.7. **Some relations between these polynomials.**

(1) In the critical case $I = \emptyset$ we have $H = M = N$, $b_x = c_x = d_x$ and $h_{y,x} = m_{y,x} = n_{y,x}$. Thus the theory of parabolic Kazhdan-Lusztig polynomials contains the theory of Kazhdan-Lusztig polynomials.

(2) If I is finitary (i.e. W_I is finite) then Deodhar [Deo87] proves that the m polynomials are instances of Kazhdan-Lusztig polynomials. More precisely, he proves that if w_0 is the longest element of W_I then $m_{y,x} = h_{w_0 y, w_0 x}$. Moreover, M is a sub-H-module of H compatible with the duality.

This result was expected. Parabolic Kazhdan-Lusztig polynomials calculate (and this is their main reason to exist) the dimensions of the intersection cohomology modules of Schubert varieties in G/P where G is a Kac-Moody group and P is a standard parabolic. Kazhdan-Lusztig polynomials calculate those dimensions in the case of the flag variety G/B. When G is a semi-simple or affine Kac-Moody group (and thus the parabolic subgroup of the Weyl group of G corresponding to P is finite) one problem reduces to the other, because one has a smooth fibration $G/B \to G/P$.

(3) For arbitrary I and $x, y \in {}^I W$, Dehodar [Deo87] proved the formula

$$n_{y,x} = \sum_{z \in W_I} (-v)^{l(z)} h_{zy,x}.$$

This fact follows from the fact that, if π is the obvious surjection $\pi : H \twoheadrightarrow N$ and $w = xy$ is the decomposition with $x \in W_I$ and $y \in {}^I W$, then

$$\pi(h_{xy}) = (-v)^{l(x)} n_y.$$

So, summarizing, M is sometimes a good sub-object and N is always a good quotient of H (seen as an H-module).

(4) If $I \subseteq J$, then $n^I_{y,x} \leq n^J_{y,x}$ (where $\leq$ denotes coefficientwise inequality). This is known as *Brenti's monotonicity conjecture* and it is due to Francesco Brenti. We prove it in this paper (see Corollary 8.4) as a corollary of our main theorem.

3. The categories $\mathcal{H}$, $\mathcal{N}$ and ${}_Q\mathcal{N}$

In this section we define the Hecke category (denoted by $\mathcal{H}$), the diagrammatic anti-spherical category (denoted by $\mathcal{N}$) and a localization (denoted by ${}_Q\mathcal{N}$). For the Hecke category we follow the exposition given in [HW, §2.5-2.7].

3.1. **Realizations.** Recall that a realization, as defined in [EW16, §3.1]) consists of a commutative ring $\Bbbk$ and a free and finitely generated $\Bbbk$-module $\mathfrak{h}$ together with subsets

$$\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$$

of "roots" and "coroots" such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ for all $s \in S$ and such that the formulas

$$s(v) := v - \langle \alpha_s, v\rangle \alpha_s^\vee \qquad \text{for } s \in S \text{ and } v \in \mathfrak{h},$$

define an action of W on $\mathfrak{h}$.

Unless otherwise stated we will assume in this paper that $\Bbbk = \mathbb{R}$ and that $\mathfrak{h}$ is the geometric representation of W, i.e. $\mathfrak{h} = \bigoplus_{s\in S} \mathbb{R}\alpha_s^\vee$ and the elements $\{\alpha_s\} \subset \mathfrak{h}^*$ are defined by the equations

$$\langle \alpha_t^\vee, \alpha_s \rangle = -2\cos(\pi/m_{st}) \tag{3.1}$$

(by convention $m_{ss} = 1$ and $\pi/\infty = 0$). Note that the subset $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ is linearly independent if and only if W is finite.

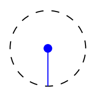
Remark 3.1. One can consider other realizations but they may introduce extra subtleties. For example one may not have the Parabolic Property. That is why we make the assumptions above.

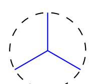
Let $R = S(\mathfrak{h}^*)$ be the ring of regular functions on $\mathfrak{h}$ or, equivalently, the symmetric algebra of $\mathfrak{h}^*$ over $\Bbbk$. We see R as a graded $\Bbbk$-algebra by declaring $\deg \mathfrak{h}^* = 2$. The action of W on $\mathfrak{h}^*$, extends to R by functoriality. For any $s \in S$, let $\partial_s : R \to R[-2]$ be the *Demazure operator* defined by the formula

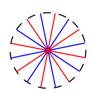
$$\partial_s(f) = \frac{f - sf}{\alpha_s}.$$

In [EW16, §3.3] it is proved that this is well defined under our assumptions.

3.2. **Towards the morphisms in $\mathcal{H}_{\mathrm{BS}}$.** An *$S$-graph* is a finite, planar, decorated graph with boundary properly embedded in the planar strip $\mathbb{R} \times [0, 1]$. Its edges are colored by S. The vertices in this graph are of 3 types:

(1) univalent vertices ("dots"): 

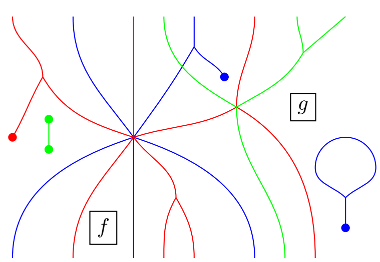
(2) trivalent vertices: 

(3) $2m_{rb}$-valent vertices: 

We require that there are exactly $2m_{rb} < \infty$ edges originating from the vertex. They alternate in color between two different elements $r, b \in S$ around the vertex. The pictured example has $m_{rb} = 8$.

Additionally any S-graph may have its regions (the connected components of the complement of the graph in $\mathbb{R} \times [0, 1]$) decorated by boxes containing homogenous elements of R.

The following is an example of an S-graph with $m_{b,r} = 5$, $m_{b,g} = 2$, $m_{g,r} = 3$:



where f and g are homogeneous polynomials in R.

The *degree* of an S-graph is the sum over the degrees of its vertices and boxes. Each box has degree equal to the degree of the corresponding element of R. The vertices have degrees given by the following rule: dots have degree 1, trivalent vertices have degree -1 and $2m$-valent vertices have degree 0. For example, the degree of the S-graph above is

$$-1 + 1 - 1 + 1 - 1 - 1 + 1 + 1 + \deg f + \deg g = \deg f + \deg g.$$

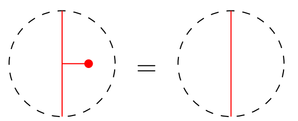
The intersection of an S-graph with $\mathbb{R} \times \{0\}$ (resp. with $\mathbb{R} \times \{1\}$) is a sequence of colored points called *bottom boundary* (resp. *top boundary*). In our example, the bottom (resp. top) boundary of the S-graph is (b, r, b, r, r, b, g, r) (resp. (r, b, r, g, b, r, g, g)).

3.3. **Relations in $\mathcal{H}_{\mathrm{BS}}$.** Let us define the Hecke category. In this section we will give a summary of the central result of [EW16].

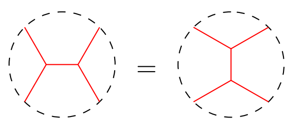
We define $\mathcal{H}_{\mathrm{BS}}$ as the monoidal category with objects sequences $\underline{w}$ in S. If $\underline{x}$ and $\underline{y}$ are two such sequences, we define $\mathrm{Hom}_{\mathcal{H}_{\mathrm{BS}}}(\underline{x}, \underline{y})$ as the free R-module generated by isotopy classes of S-graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo the local relations below. Hom spaces are graded by the degree of the graphs (all the relations below are homogeneous). The structure of this monoidal category is given by horizontal and vertical concatenation of diagrams.

In what follows, the rank of a relation is the number of colors involved in the relation. We use the color red for r and blue for b.

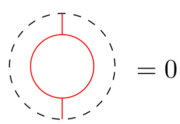
3.3.1. *Rank 1 relations. Frobenius unit:*

(3.2) 

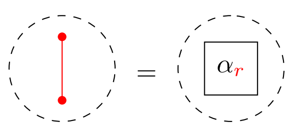
Frobenius associativity:

(3.3) 

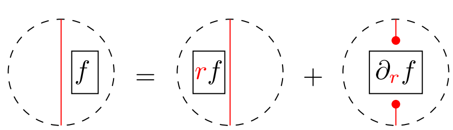
Needle relation:

(3.4) 

Barbell relation:

(3.5) 

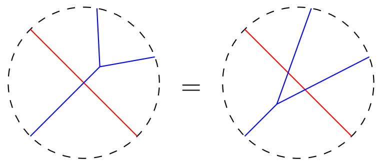
Nil Hecke relation:

(3.6) 

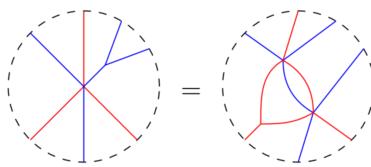
(See §3.1 for the definition of ∂_r.)

3.3.2. *Rank 2 relations. Two-color associativity:* We give the first three cases i.e. $m_{rb} = 2, 3, 4$. It is not hard to guess this relation for arbitrary m_{rb} (see [Eli16, 6.12] for details).

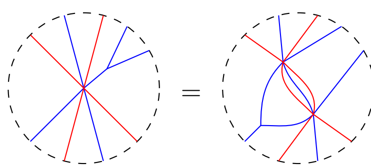
$m_{rb} = 2$ (type $A_1 \times A_1$):



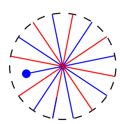
$m_{rb} = 3$ (type A_2):



$m_{rb} = 3$ (type B_2):

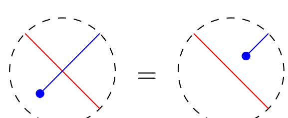


Elias' Jones-Wenzl relation: This relation expresses a dotted $2m_{rb}$-vertex

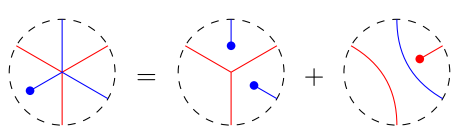


as a linear combination over R of diagrams consisting only of trivalent vertices and dots (no $2m_{rb}$-valent vertices). We present again the first three cases i.e. $m_{rb} = 2, 3, 4$ (this time it is not easy to guess the general form, see [Eli16, 6.13] for all the details).

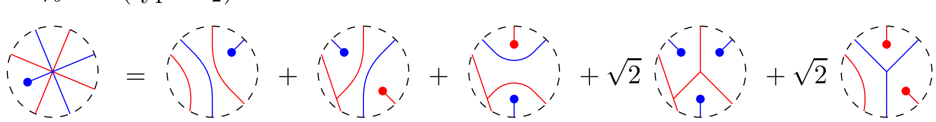
$m_{rb} = 2$ (type $A_1 \times A_1$):



$m_{rb} = 3$ (type A_2):



$m_{rb} = 4$ (type B_2):



In all the cases one obtains the other Jones-Wenzl relation by swapping red and blue.

3.3.3. *Rank 3 relations.* We will not repeat the definition of the Zamolodchikov relations here, and instead refer the reader to [EW16, §1.4.3]. This concludes the definition of $\mathcal{H}_{\mathrm{BS}}$.

3.4. **The categories $\mathcal{H}$ and $\mathcal{H}_I$.** If $M = \bigoplus_i M^i$ is a $\mathbb{Z}$-graded object, we denote by $M(1)$ its grading shift, i.e. $M(1)^i = M^{i+1}$. Given an additive category $\mathcal{A}$ we denote by $[\mathcal{A}]$ its split Grothendieck group. If in addition $\mathcal{A}$ has homomorphism spaces enriched in graded vector spaces we denote by $\mathcal{A}^{\oplus}$ its additive graded envelope. That is, objects are formal finite direct sums $\bigoplus a_i(m_i)$ for certain objects $a_i \in \mathcal{A}$ and "grading shifts" $m_i \in \mathbb{Z}$. Homomorphism spaces in $\mathcal{A}^{\oplus}$ are given by

$$\mathrm{Hom}_{\mathcal{A}^{\oplus}}(\bigoplus a_i(m_i), \bigoplus a'_j(m'_j)) := \bigoplus \mathrm{Hom}(a_i, a'_j)(m'_j - m_i).$$

We denote by $\mathcal{A}^{\oplus,0}$ the category with the same objects as $\mathcal{A}^{\oplus}$ but with homomorphism spaces given by the degree zero morphisms in $\mathcal{A}^{\oplus}$:

$$\mathrm{Hom}_{\mathcal{A}^{\oplus,0}}(b, b) := \mathrm{Hom}_{\mathcal{A}^{\oplus}}(b, b')^0.$$

Both $\mathcal{A}^{\oplus}$ and $\mathcal{A}^{\oplus,0}$ are equipped with a grading shift functor $b \mapsto b(1)$ given on objects by $\bigoplus a_i(m_i) \mapsto \bigoplus a_i(m_i + 1)$. Of course $\mathcal{A}^{\oplus}$ is recoverable from $\mathcal{A}^{\oplus,0}$ and the grading shift functor (1). Finally we define $\mathcal{A}^e$ to be the Karoubian envelope of the $\mathcal{A}^{\oplus,0}$. In this setting, given objects $b, b' \in \mathcal{A}^e$ we abbreviate:

$$\mathrm{Hom}(b, b') := \mathrm{Hom}_{\mathcal{A}^e}(b, b'),$$
$$\mathrm{Hom}^{\bullet}(b, b') := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(b, b'(m)).$$

By definition the Hecke category $\mathcal{H}$ is $\mathcal{H}^e_{\mathrm{BS}}$. If $I \subset S$, we define $\mathcal{H}_I := \mathcal{H}^e_{\mathrm{BS},I}$, where $\mathcal{H}_{\mathrm{BS},I}$ is the diagrammatic category obtained by replacing S by I in the definition. (That is, all diagrams in $\mathcal{H}_{\mathrm{BS},I}$ are only allowed to be colored by elements of I.)

3.5. **Basic facts about $\mathcal{H}$.** Let us recall some terminology and notations from [EW16]. A *subsequence* of an expression $\underline{x} = s_1 s_2 \dots s_m$ is a sequence $\pi_1\pi_2 \dots \pi_m$ such that $\pi_i \in \{e, s_i\}$ for all $1 \le i \le m$. Instead of working with subsequences, we work with the equivalent datum of a sequence $\mathbf{e} = \mathbf{e}_1\mathbf{e}_2 \dots \mathbf{e}_m$ of 1's and 0's giving the indicator function of a subsequence, which we refer to as a *01-sequence.*

The *Bruhat stroll* is the sequence $x_0 = e, x_1, \dots, x_m$ defined by

$$x_i := s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2} \dots s_i^{\mathbf{e}_i}$$

for $0 \le i \le m$. We call x_i the i^{th}*-point* and x_m the *end-point* of the Bruhat stroll. We denote x_m by $\underline{x}^{\mathbf{e}}$. Alternatively, we will say that a subsequence $\mathbf{e}$ of $\underline{x}$ *expresses* the end-point $\underline{x}^{\mathbf{e}}$.

Let $\mathbf{e}$ and $\mathbf{f}$ be two 01-sequences of $\underline{x} = s_1 s_2 \cdots s_m$ and let their corresponding Bruhat strolls be $x_0, x_1, \dots, x_m$ and $y_0, y_1, \dots, y_m$. We say that $\mathbf{e} \ge \mathbf{f}$ in the *path dominance order* if $x_i \ge y_i$ for all $0 \le i \le m$. We define the *double path dominance order* (a partial order) on pairs $(\mathbf{e}, \mathbf{f})$, where $(\mathbf{e}_1, \mathbf{f}_1) \le (\mathbf{e}_2, \mathbf{f}_2)$ if $\mathbf{e}_1 \le \mathbf{e}_2$ and $\mathbf{f}_1 \le \mathbf{f}_2$.

Light leaves and *Double leaves* for Soergel bimodules were introduced in [Lib08b] and [Lib15]. They give bases, as R-modules of the Hom spaces between Bott-Samelson bimodules. We recommend reading the paper [Lib15] in order to get used to these combinatorial objects and to read § 6.1–6.3 of [EW16], where these bases are explained diagrammatically.

In [EW16, Definition 6.24] the authors define a character map $\mathrm{ch} : [\mathcal{H}] \to H$ and in [EW16, Corollary 6.27] they prove that the map ch is an isomorphism. This is the reason why we call $\mathcal{H}$ the Hecke category.

If $p = \sum_j a_j v^j \in \mathbb{Z}_{>0}[v^{\pm 1}]$, we denote

$$p \cdot M = \oplus_j M(j)^{\oplus a_j}.$$

Following Soergel's classification of indecomposable Soergel bimodules, in [EW16, Theorem 6.26] the authors prove that the indecomposable objects in $\mathcal{H}$ are indexed by W modulo shift, and they call B_w the indecomposable object corresponding to $w \in W$. It happens that the object B_s is the sequence with one element $(s) \in \mathcal{H}$. Because of this, if $\underline{w} = (s, r, \cdots, t)$ we will sometimes denote by $B_{\underline{w}} := B_s B_r \cdots B_t$ the element $\underline{w} \in \mathcal{H}$.

The main result of [EW14] is that $\mathrm{ch}([B_w]) = b_w$. (More precisely, to obtain this result one must combine the equivalence between $\mathcal{H}$ and the category of Soergel bimodules proved in [EW16] with the main results of [EW14] and [Lib08a].) Thus the indecomposable objects in $\mathcal{H}$ categorify the Kazhdan-Lusztig basis. We will use this important fact repeatedly below.

3.6. **The anti-spherical category $\mathcal{N}$.** Fix a subset $I \subset S$. We define the *Bott-Samelson anti-spherical category* $\mathcal{N}_{\mathrm{BS}}$ to be the categroy $\mathcal{H}_{\mathrm{BS}}$ quotiented by the ideal of all objects indexed by I-sequences. The *anti-spherical category* $\mathcal{N}$ is the graded additive Karoubian completion of $\mathcal{N}_{\mathrm{BS}}$, i.e. $\mathcal{N}_{\mathrm{BS}}^e := \mathcal{N}$. For $x \in {}^I W$ we call D_x the image of B_x in the anti-spherical category $\mathcal{N}$.

We define the category $\mathcal{N}'$ to be the categroy $\mathcal{H}$ quotiented by the ideal of all objects $B_x \in \mathcal{H}$, with $x \notin {}^I W$

Proposition 3.2. *We have an equivalence of categories $\mathcal{N} \cong \mathcal{N}'$.*

Proof. Consider the monoidal functor $\mathcal{F}_1 : \mathcal{H}_{\mathrm{BS}} \to \mathcal{N}'$ defined as the composition of the inclusion functor $\mathcal{H}_{\mathrm{BS}} \hookrightarrow \mathcal{H}$ with the canonical projection $\mathcal{H} \to \mathcal{N}'$.

Let $s \in S$ and $x \in W$. If we expand $b_s b_x = \sum_{y<sx} m_y b_y$, in terms of the Kazhdan-Lusztig basis, we have that $m_y \in \mathbb{N}$ and that $m_y \neq 0 \Rightarrow sy < y$. This means that if $\underline{w}$ is an I-sequence, say $\underline{w} = (s, s_1 \ldots, s_n)$ with $s \in I$, when we decompose the sequence $(s_1 \ldots, s_n)$ into indecomposable summands $\oplus_z p_z \cdot B_z$ (with $p_z \in \mathbb{Z}_{>0}[v^{\pm 1}]$) every z appearing in this sum is such that $sz < z$, thus $z \notin {}^I W$, and by definition they are zero in $\mathcal{N}'$. So the functor $\mathcal{F}_1$ factors through the ideal generated by all I-sequences, giving a functor $\mathcal{F}_2 : \mathcal{N}_{\mathrm{BS}} \to \mathcal{N}'$. The category $\mathcal{N}'$ is clearly idempotent complete, so the functor $\mathcal{F}_2$ gives us a functor $\mathcal{F}_3 : \mathcal{N} \to \mathcal{N}'$.

We will now prove that $\mathcal{F}_3$ is an equivalence of categories by finding an inverse equivalence $\mathcal{G}_3 : \mathcal{N} \to \mathcal{N}'$. Let $\mathcal{G}_1 : \mathcal{H}_{\mathrm{BS}} \to \mathcal{N}_{\mathrm{BS}}$ be the canonical projection. This functor lifts to a functor between the corresponding Karoubian completions $\mathcal{G}_2 : \mathcal{H} \to \mathcal{N}$. This functor is clearly zero on any $B_x \in \mathcal{H}$ such that $x \notin^I W$ because any such element is a summand of an I-sequence. This gives us a functor $\mathcal{G}_3 : \mathcal{N}' \to \mathcal{N}$ that is clearly an inverse equivalence to $\mathcal{F}_3$. □

3.7. ${}_Q\mathcal{N}$**: a localization of $\mathcal{N}$.** We will see in this section that a certain localized version of $\mathcal{N}$ is as simple as it can be. Thus the situation is optimal for $\mathcal{N}$, as we knew that it is for $\mathcal{H}$ (see [EW16]).

For $I \subseteq S$, define the *coinvariant ring* $R_I := R/\langle \alpha_s | s \in I\rangle$. It is the largest quotient on which the parabolic group W_I acts trivially. If A is either the ring R or the ring R_I, we use the notation $A(\frac{1}{\Phi_I^c})$ for the localization of A by all the roots that are not in Φ_I. In formulas, $A(\frac{1}{\Phi_I^c}) = A[\alpha^{-1}|\alpha \notin \Phi_I]$. We define $Q_I := R_I(\frac{1}{\Phi_I^c})$ (i.e. "kill I and invert the rest"). Finally, we define the object of study of the following section ${}_Q\mathcal{N} := (Q_I \otimes_{R_I} \mathcal{N}_{\mathrm{BS}})^e$. The category ${}_Q\mathcal{N}$ is still a right $\mathcal{H}$-module.

Theorem 3.3. *In ${}_Q\mathcal{N}$ there is a set of objects $\{Q_x\}_{x\in {}^IW}$ satisfying the following properties.*

1. $Q_{\mathrm{id}} = N_{\mathrm{id}}$ *(the image in ${}_Q\mathcal{N}$ of the empty sequence in $\mathcal{H}_{\mathrm{BS}}$).*
2. $Q_x f = x(f)Q_x$ for $f \in R$.
3. *For all $x \in {}^IW$ we have* $Q_xB_s \cong \begin{cases} Q_x \oplus Q_{xs} & \text{if } xs \in {}^IW, \\ 0 & \text{if } xs \notin {}^IW. \end{cases}$
4. *For all $x, y \in {}^IW$ we have* $\mathrm{Hom}(Q_x, Q_y) = \delta_{x,y}Q_I \cdot \mathrm{id}_{Q_x}$ *(where $\delta_{x,y}$ is the Kronecker delta).*
5. *Any object in ${}_Q\mathcal{N}$ is isomorphic to a direct sum of Q_x's.*

4. Proof of parts (1), (2) and (3) of Theorem 3.3

Lemma 4.1. *For each $x \in {}^IW$ and for each reduced expression $\underline{x}$ of x we can define an element $Q_{\underline{x}}$ that satisfies the equation*

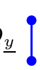
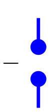
$$Q_{\underline{x}}f = x(f)Q_{\underline{x}}. \tag{4.1}$$

Proof. Let us define $Q_{\underline{x}}$ by induction in the length of $\underline{x}$. If this length is zero, we define $Q_\emptyset = N_{\mathrm{id}}$. Let us suppose that we have defined $Q_{\underline{y}}$ for all $\underline{y}$ reduced expressions of all elements $y \in {}^IW$ with $l(\underline{y}) < l(\underline{x})$ and that equation (4.1) holds for all such $Q_{\underline{y}}$.

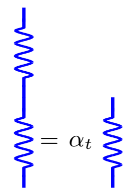
Write $\underline{x} = (s_1s_2\cdots s_{m-1})s$ and set $\underline{y} = s_1s_2\cdots s_{m-1}$. Recall that $x, y \in {}^IW$. We want to define $Q_{\underline{x}}$ in such a way that it satisfies equation (3) of the theorem, i.e. we want

$$Q_{\underline{y}}B_s \cong Q_{\underline{y}} \oplus Q_{\underline{x}}, \tag{4.2}$$

so we will produce the two respective projectors $e_{\underline{y}}, e_{\underline{x}} \in \mathrm{End}_{Q\mathcal{N}}(Q_{\underline{y}}B_s)$.

- By the induction hypothesis, we have
$$Q_{\underline{y}}\alpha_s = y(\alpha_s)Q_{\underline{y}}$$
and by the Parabolic Property 2.3 we have that $y(\alpha_s) \notin \Phi_I$, hence
$Q_{\underline{y}}$  $\in \mathrm{End}_{Q\mathcal{N}}(Q_{\underline{y}}B_s)$
is an isomorphism. Then we define the idempotent
$e_{\underline{y}} := y(\alpha_s)^{-1}Q_{\underline{y}}$  $\in \mathrm{End}_{Q\mathcal{N}}(Q_{\underline{y}}B_s)$
- Let us define the *squiggly morphism*. For $t \in S$,
 $:= \alpha_t$  $-$  $\in \mathrm{End}_{Q\mathcal{N}}(B_t)$

It is easy to see that



Thus we have that the morphism

$$e_{\underline{x}} := y(\alpha_s)^{-1}Q_{\underline{y}}$$  $$\in \mathrm{End}_{Q\mathcal{N}}(Q_{\underline{y}}B_s)$$

is an idempotent.

We define $Q_{\underline{x}}$ to be the image of $e_{\underline{x}}$ in ${}_Q\mathcal{N}$ (recall that by definition, elements in ${}_Q\mathcal{N}$ are pairs (n, e) with $n \in \mathcal{N}_{\mathrm{BS}}$ and $e \in \mathrm{End}(n)$ an idempotent).

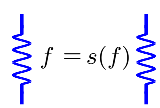
We now check that with these definitions we obtain equation (4.2): it is easy to check that $e_{\underline{x}}e_{\underline{y}} = e_{\underline{y}}e_{\underline{x}} = 0$. So we just need to check that the image of $e_{\underline{y}}$ is isomorphic to $Q_{\underline{y}}$. This isomorphism is given by the map

$$Q_{\underline{y}}$$  $$: Q_{\underline{y}}B_s \to Q_{\underline{y}}$$

with inverse given by the morphism

$$y(\alpha_s)^{-1}Q_{\underline{y}}$$  $$: Q_{\underline{y}} \to Q_{\underline{y}}B_s$$

Now are now ready to finish the proof of Lemma 4.1. By induction hypothesis, it is enough to prove that



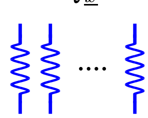
but this follows directly from the Nil Hecke relation and the definition of Demazure operators. ☐

4.1. We use the abbreviation *rex* for the sentence "reduced expression". The *rex graph* of $x \in W$ is the finite graph with vertices the rex of x and where two rex are connected by an edge if they differ by a single application of a braid relation.

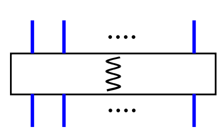
The following lemma says that Q_x is well defined up to isomorphism.

Lemma 4.2. *Any path $\underline{x} \to \underline{x}'$ in the rex graph of an element $x \in W$ induces an isomorphism from $Q_{\underline{x}}$ to $Q_{\underline{x}'}$.*

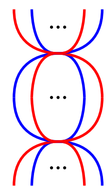
Proof. Let $\underline{x}$ be a rex of $x \in W$. If we define $N_{\underline{x}} := N_{\mathrm{id}} \cdot B_{\underline{x}}$ (i.e. the image in ${}_Q\mathcal{N}$ of $B_{\underline{x}}$), then we have that by definition $Q_{\underline{x}}$ is the image of the projector

$$\frac{1}{p}$$  $$\in \mathrm{End}(N_{\underline{x}})$$

where p is some polynomial in R. We will sometimes denote this projector by a squiggly box



Claim 4.3. *If $m_{rb} < \infty$ and one expresses the composition of two $2m_{rb}$-valent vertices*



in terms of the double leaves, we obtain the identity map plus terms with at least one dot on strands $2, 3, \ldots, (m_{rb} - 1)$ on top and bottom.

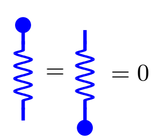
Proof. By the construction of double leaves between reduced expressions, if some double leaf is not the identity, then there are some dots on the top and on the bottom leaves. If there was a double leaf with a dot, say in the top and in the right-most position, this would induce a map (just by ignoring that dot and taking the rest) in

$$\mathrm{Hom}(\underbrace{B_r B_b B_r \cdots}_{m_{rb}}, \underbrace{B_r B_b B_r \cdots}_{m_{rb}-1}).$$

This map would be of degree -1 (because the $2m_{rb}$-valent morphism has degree zero and the dot has degree 1), but there is no degree -1 morphism in that Hom space by easy considerations in the Hecke algebra (using Soergel's formula for the graded degrees of the Hom spaces). For another proof of this claim see [Eli16, Claim 6.17]. □

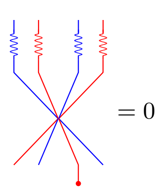
It is trivial to check the relation

(4.3)



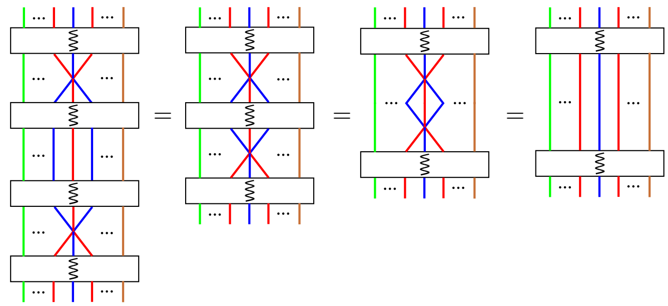
By the Jones-Wenzl relation, any composition of a dot, a $2m_{rb}$-valent vertex and the projection to $Q_{\underline{x}}$ is always zero:

(4.4)



To finish the proof of Lemma 4.2 it is enough to prove that if $\underline{x}$ and $\underline{x}'$ differ by a single braid relation, then the corresponding $2m_{rb}$-valent morphism from $N_{\underline{x}}$ to

$N_{\underline{x}'}$ induces an isomorphism from $Q_{\underline{x}}$ to $Q_{\underline{x}'}$. To see this, just follow the equalities



(In this picture we consider $m_{rb} = 3$, with r red and b blue). The first equality is because the squiggly box is an idempotent. The second one is because of the definition of the squiggly morphism combined with equation (4.4) (one has to take all the α_r and α_b that appear to the leftmost position, going from right to left, always using Equation 4.1). The third equation is a direct consequence of Claim 4.3 (be aware that we are not yet using its full power) and Equation (4.3). This concludes the proof of Lemma 4.2. □

Let $\underline{x}$ and $\underline{x}'$ be two rex of the same element in W. It is proved in [EW16] that any two paths in the rex graph from $\underline{x}$ to $\underline{x}'$ give the same morphism in $\mathcal{H}$ modulo lower terms (i.e. modulo morphisms which factor through $\underline{y}$ for a sequence $\underline{y}$ strictly shorter than $\underline{x}$). As lower terms die after projecting to $Q_{\underline{x}'}$ (again, when expressed as linear combination of double leaves they always have dots), the elements $Q_{\underline{x}}$ and $Q_{\underline{x}'}$ are canonically isomorphic. We define Q_x to be the limit of the transitive system defined by the $Q_{\underline{x}}$ with $\underline{x}$ reduced expressions of x. With this definition, the first part (1) of Theorem 3.3 is given by definition. The second part (2) is Equation 4.1.

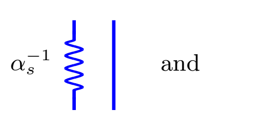
4.2. Now we attack Part 3 of the theorem.

Lemma 4.4. *Let $x \in {}^I W$. If $xs \in {}^I W$ and $xs < x$, we have that $Q_x B_s \cong Q_x \oplus Q_{xs}$.*

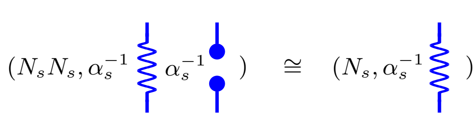
Proof. We have already proved (Equation 4.2) that $Q_{xs} B_s \cong Q_x \oplus Q_{xs}$, so we just need to prove that

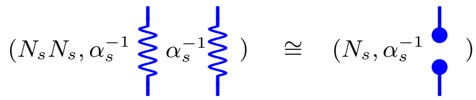
$$Q_{xs} B_s \cong Q_x B_s.$$

By choosing $\underline{x}$ a rex of x such that the rightmost term is s, we reduce the problem to prove that (when α_s is invertible in the left of $N_s N_s$), the following two morphisms have isomorphic images

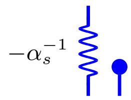


This reduces to check the two isomorphisms in ${}_Q\mathcal{N}$ (where $s \notin \Phi_I$):

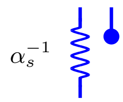




The first one is given by the map



and the second one is given by the map



To prove that these are inverse isomorphisms is a fun diagrammatical exercise. □

Lemma 4.5. *Let $x \in {}^I W$. If $xs \notin {}^I W$, we have that $Q_x B_s = 0$.*

Proof. We will prove this Lemma by induction on the length of x. When $l(x) = 0$ we have that $x = \mathrm{id}$, $s \in I$ and the lemma is clear. So we suppose that $x \neq \mathrm{id}$ and that the lemma is true for all the lengths lesser than $l(x)$. As $x \in {}^I W$ and $xs \notin {}^I W$, we have that there is a rex $\underline{x}$ of x such that

$$\underline{x} = \begin{cases} \underline{x}_{\min}(\underbrace{st\ldots t}_{m_{st}-1}) \text{ if } m_{st} \text{ is odd,} \\ \underline{x}_{\min}(\underbrace{ts\ldots t}_{m_{st}-1}) \text{ if } m_{st} \text{ is even.} \end{cases}$$

It is not difficult to prove that if w_0 is the longest element of the parabolic subgroup $\langle s, t\rangle$ then

$$w_0(\alpha_s) = \begin{cases} -\alpha_t \text{ if } m_{st} \text{ is odd,} \\ -\alpha_s \text{ if } m_{st} \text{ is even.} \end{cases}$$

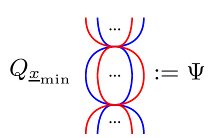
Thus we obtain

$$x(\alpha_s) = \begin{cases} x_{\min}(\alpha_t) \text{ if } m_{st} \text{ is odd,} \\ x_{\min}(\alpha_s) \text{ if } m_{st} \text{ is even.} \end{cases}$$

As $x(\alpha_s) \in \Phi_I$ (because of the Parabolic Property 2.3), by the same property we deduce that $x_{\min}s \notin {}^I W$ when m_{st} is even and $x_{\min}t \notin {}^I W$ when m_{st} is odd. By induction we have that $Q_{x_{\min}} B_t = 0$ if m_{st} is odd and $Q_{x_{\min}} B_s = 0$ if m_{st} is even. These annulations are enough to finish the proof of this lemma because by construction

$$Q_{\underline{x}} B_s \text{ is a direct summand of } \begin{cases} Q_{\underline{x}_{\min}}(\underbrace{B_s B_t \ldots B_s}_{m_{st}}) \text{ if } m_{st} \text{ is odd,} \\ Q_{\underline{x}_{\min}}(\underbrace{B_t B_s \ldots B_s}_{m_{st}}) \text{ if } m_{st} \text{ is even.} \end{cases}$$

and in both cases the morphism



induces the identity on $Q_{\underline{x}}B_s$ (by Claim 4.3), but the aforementioned annulations say that $\Psi = 0$, because it factors through 0. □

Thus, we have finished the proof of Part 3 of Theorem 3.3. Equation 4.2 gives us the case where $xs \in {}^IW$ and $xs > x$. Lemma 4.4 gives us the case where $xs \in {}^IW$ and $xs < x$ and finally, Lemma 4.5 give us the case where $xs \notin {}^IW$.

5. The infinite twist

In this section we show that it makes sense to raise the Rouquier complex associated to the longest element in a finite parabolic subgroup to an infinite power. This idea (for the symmetric group) first emerged in the link homology literature [DGR06, Roz14, Cau15, Cau16].

In this section we consider I to be finitary. Let w_I be the longest element of W_I. We denote by $K = K(\mathcal{H})$ the unbounded homotopy category of the Hecke category. $K^{<m}$ (resp. $K^{\geq m}$ etc.) the full subcategory of complexes whose minimal complex (i.e. complexes with no contractible direct summands) is zero in degrees above and including m (resp. below m, etc.) We denote by $\mathcal{H}_{>\mathrm{id}} := \langle B_x \mid x \neq \mathrm{id}\rangle_{\oplus,(m)}$.

Lemma 5.1. *$\mathcal{H}_{>\mathrm{id}}$ is an ideal of the tensor category $\mathcal{H}$.*

Proof. It is enough to prove that the $\mathbb{Z}[v^{\pm 1}]$ module generated by the set $\{b_x\}_{x\neq \mathrm{id}}$ is an ideal of H. This is an easy consequence of the multiplication formula for $b_x b_s$ in the Hecke algebra. □

Lemma 5.2. *If $X \in K$ satisfies $XB_s \in K^{\leq n}$ for all $s \in S$ then $XB \in K^{\leq n}$ for all $B \in \mathcal{H}_{>\mathrm{id}}$.*

Proof. Choose $w \in W$. Then B_w is a summand of $B_{\underline{w}}$. Now, if $w \neq \mathrm{id}$ then $XB_{\underline{w}} \in K^{\leq n}$, because $K^{\leq n}$ is stable under multiplication by objects of $\mathcal{H}$. Hence $XB_w \in K^{\leq n}$, because $K^{\leq n}$ is idempotent complete [Sch, Thm 3.1]. The lemma now follows because any element of $\mathcal{H}_{>\mathrm{id}}$ is a direct sum of shifts of B_w with $w \neq \mathrm{id}$. □

The following lemma is a direct consequence of the previous two lemmas.

Lemma 5.3. *If D is an element of $K^{<0}(\mathcal{H}_{>\mathrm{id}})$ and $X \in K(\mathcal{H})$ satisfies that $XB_s \in K^{\leq n}(\mathcal{H})$ for all $s \in S$, then $XD \in K^{\leq n-1}$.*

Remark 5.4. We will apply this lemma when S is replaced by I and $\mathcal{H}$ by $\mathcal{H}_I$

Let $F_s := B_s(-1) \to R \in K$ with R in degree zero (note that this is not the usual convention). Given any element $w \in W$ we set

$$F_w = F_{\underline{w}} := F_s F_t \dots F_u$$

where $\underline{w} := st \dots u$ is a reduced expression for w. It is a result of Rouquier [Rou06] that $F_w \in K$ does not depend on the reduced expression chosen for w. A direct calculation and the isomorphism

$$B_s B_s \cong B_s(1) \oplus B_s(-1) \tag{5.1}$$

show that

$$F_s B_s \cong B_s(-2)[1]. \tag{5.2}$$

Lemma 5.5. *We have a distinguished triangle*

$$R \to F_{w_I} \to C \xrightarrow{[1]} \tag{5.3}$$

with $C := Cone(R \to F_{w_I}) \in K^{<0}((\mathcal{H}_I)_{>\mathrm{id}})$.

Proof. The lemma is immediate from the definitions, and the fact that $\mathcal{H}_{>\mathrm{id}}$ is an ideal. □

Lemma 5.6. *For all* $s \in I$*, we have*

$$F_{w_I}^m B_s \in K^{\leq -m}.$$

Proof. We prove this lemma by induction on m. It is clear for $m = 0$. By definition of F_{w_I} and (5.2), for any $X \in K(\mathcal{H}_I)$,

$$XF_{w_I}B_s = XF_{w_I s}F_sB_s = XF_{w_I s}B_s(-2)[1].$$

It is clear that $F_{w_I s}B_s \in K^{\leq 0}(\mathcal{H}_{>\mathrm{id}})$. Thus

$$F_{w_I s}B_s(-2)[1] \in K^{<0}((\mathcal{H}_I)_{>\mathrm{id}})$$

and we can use Lemma 5.3 to conclude that, if $X = F_{w_I}^m$ and using the induction hypothesis, then $XF_{w_I s}B_s(-2)[1]$ belongs to $K^{\leq -m-1}$, thus proving the statement. □

Proposition 5.7. *The first morphism of* (5.3) *gives a tower*

$$0 \to R \xrightarrow{a_0} F_{w_I} \xrightarrow{a_1} F_{w_I}^2 \xrightarrow{a_2} F_{w_I}^3 \xrightarrow{a_3} \dots$$

with $a_m := \mathrm{id}_{F_{w_I}^m} \otimes a_0$*. If all the complexes in the sequence are minimal then the morphism* a_m *is an isomorphism in degrees* $\geq -m$ *(i.e.* $Cone(a_m) \in K^{\leq -m-1}$*).*

Proof. We prove this proposition by induction on m. It is clear for $m = 0$. Now we do the inductive step. Suppose that $X \in K(\mathcal{H}_I)$ satisfies $XB_s \in K^{\leq -m}$ for all $s \in I$. By Lemma 5.5, and as a distinguished triangle in K tensored by an object is still a distinguished triangle (K is a tensor triangulated category), we have

$$X \to XF_{w_I} \to XC \xrightarrow{[1]}$$

This implies that $\mathrm{cone}(X \to XF_{w_I})) \cong XC$, (see e.g. [GM03, IV.1.4 Corollary a)]). But as $C \in K^{<0}((\mathcal{H}_I)_{>\mathrm{id}})$, we have $XC \in K^{\leq -m-1}$, by Lemma 5.3. By Lemma 5.6 we can use $X = F_{w_I}^m$, thus proving the proposition. □

As a_i is an isomorphism in degrees $\geq -i$, this complex has a well-defined limit:

$$F_I^\infty := \lim_{i \to \infty} F_{w_I}^i \in K^-(\mathcal{H}).$$

We refer to F_I^∞ as the *infinite twist.*

From Proposition 5.7 and Lemma 5.6, we also deduce the important vanishing:

$$F_I^\infty B_s = 0 \quad \text{for all } s \in I. \tag{5.4}$$

Remark 5.8. The "convergence" of F_I^m to F_I^∞ given by the lemmas above is quite slow. In examples the minimal complexes F_I^m appear to converge much more quickly to F_I^∞. We don't know what the optimal rate of convergence is.

6. Proof of parts (4) and (5) of Theorem 3.3

6.1. **Proof of parts (4) and (5) of Theorem 3.3 for finitary I.** In this section we will consider I to be finitary. The fact that $\mathrm{Hom}(Q_x, Q_y) = 0$ if $x \neq y$ is easy: any double leaf in $\mathrm{Hom}(B_{\underline{x}}, B_{\underline{y}})$ must contain a dot and pre-composing with the projector to Q_x and post-composing with the projector to Q_y we will obtain zero. The same argument shows that

$$\mathrm{End}(Q_x) \subseteq Q_I \cdot Q_x.$$

So the difficult step to prove part (4) of the Theorem is the following proposition.

Proposition 6.1. *For $x \in {}^I W$ we have* $\mathrm{End}(Q_x) \supseteq Q_x \cdot x^{-1}(Q_I)$

Remark 6.2. In particular, this proposition implies that $0 \neq Q_x \in {}_Q\mathcal{N}$.

Proof. Recall that with the realization we are working with, we have [EW16, Theorem 6.30] an equivalence of categories $\mathcal{B} \cong \mathcal{H}$ (where $\mathcal{B}$ is the category of Soergel bimodules). Let us call $\mathcal{B}' := R(\frac{1}{\Phi_I^c}) \otimes_R \mathcal{B}$ and $\mathcal{H}' := R(\frac{1}{\Phi_I^c}) \otimes_R \mathcal{H}$. Then we have a functor of right $\mathcal{H}$-categories

$$\mathcal{H}' \to K^-(\mathcal{B}'),$$

that at the level of objects is

$$M \to F_I^\infty M$$

(in fact, at this point in the argument we could take F_I^∞ to be any element in $K^-(\mathcal{B}')$). By equation 5.4, we have that the image of α_s and of B_s with $s \in I$ is zero, thus (by the universal property of localization) we obtain a functor of right $\mathcal{H}$-modules

$$\mathcal{J} : {}_Q\mathcal{N} \to K^-(\mathcal{B}')$$

By definition, we have that $\mathcal{J}(Q_x) = F_I^\infty Q_x$, where Q_x denotes a "standard bimodule" (see e.g. [EW16, §3.4]), so if $f \in x^{-1}(Q_I)$ is such that

$$0 = Q_x \cdot f \in \mathrm{End}(Q_x)$$

then we also have

$$\mathcal{J}(Q_x \cdot f) = F_I^\infty Q_x \cdot f = 0,$$

and we just need to prove that this implies $f = 0$. For this we need to express $F_I^\infty Q_x$ in a different way that will be easier for computations.

Consider the R-bimodule $Q'_x := R(\frac{1}{\Phi_I^c}) \otimes_R R_x$, and define the complex

$$F_{I,x}^\infty := F_I^\infty \otimes_R Q'_x \in K^-(R - \mathrm{grBim}).$$

Lemma 6.3. *In the homotopy category $K^-(R - \mathrm{grBim})$ we have the following isomorphism*

$$F_I^\infty Q_x \cong F_{I,x}^\infty$$

Proof. It is enough to prove the following equation

$$F_{I,x}^\infty B_s = \begin{cases} F_{I,x}^\infty \oplus F_{I,xs}^\infty & \text{if } xs \in {}^I W, \\ 0 & \text{if } xs \notin {}^I W. \end{cases} \tag{6.1}$$

This is because this equation (resp. part 3 of Theorem 3.3) means that $F_{I,x}^{\infty}$ (resp. Q_x) can be characterized as the first summand that does not appear in shorter expressions and thus, as

$$\mathcal{J}(Q_{\mathrm{id}}) = F_I^{\infty} \otimes_R R(\frac{1}{\Phi_I^c}) = F_{I,\mathrm{id}}^{\infty}$$

and $\mathcal{J}$ is an additive functor, by induction on the length of x we conclude.

Let us prove Equation 6.1 Let us start with the first case i.e. $xs \in {}^IW$. By the Parabolic property 2.3, we have that $x(\alpha_s) \notin \Phi_I^c$, so multiplication by α_s on the right of Q'_x is invertible. This means that

$$Q'_x B_s \cong Q'_x \oplus Q'_{xs},$$

so

$$F_{I,x}^{\infty} B_s = F_{I,x}^{\infty} \oplus F_{I,xs}^{\infty},$$

thus proving the first assertion.

Let us consider the second case, i.e. $xs \notin {}^IW$. Again by the Parabolic property 2.3, $x(\alpha_s) \in \Phi_I$. We have that $Q'_x B_s \cong R(\frac{1}{\Phi_I^c}) \otimes_R R_x B_s R_{x^{-1}} R_x$ because $R_{x^{-1}} R_x \cong R$, so we obtain the isomorphism $Q'_x B_s \cong R(\frac{1}{\Phi_I^c}) B_{s_{x(\alpha_s)}} R_x$. This gives

$$F_{I,x}^{\infty} B_s \cong F_I^{\infty} R(\frac{1}{\Phi_I^c}) B_{s_{x(\alpha_s)}} R_x,$$

so using the vanishing 5.4 we obtain $F_{I,x}^{\infty} B_s = 0$, thus proving the lemma. □

Let $f \in x^{-1}(Q_I)$ be such that $0 = F_{I,x}^{\infty} \cdot f \in K^-(R(\frac{1}{\Phi_I^c}) \otimes_R \mathcal{B})$. We just need to prove that this implies that $f = 0$. We know that

$$F_{I,x}^{\infty} \cong \cdots \to \bigoplus_{s \in I} B_s(-1) Q'_x \to Q'_x.$$

If $F_{I,x}^{\infty} \cdot f = 0$, there exists a map $g_0 : Q'_x \to \bigoplus_{s \in I} B_s(-1) Q'_x$ such that $Q'_x \cdot f = \partial g_0$. As a map from Q'_x to $B_s(-1)Q'_x$ must be a map from R to $B_s(-1)$ tensored on the right by Q'_x, it has to be a multiple of the dot morphism. Thus the Barbell relation tells us that $x(f) \in \langle \alpha_s \,|\, s \in I \rangle \in R$. But we also have that $x(f) \in Q_I$, thus $f = 0$ and we finish the proof of Proposition 6.1. □

6.2. **Proof of parts (4) and (5) of Theorem 3.3 for general I.** As earlier we fix a subset $I \subset S$. However in this section we no longer assume that I is finitary.

Theorem 6.4. *There exists a right $\mathcal{H}$-module ${}_Q\mathcal{N}^g$ with the following properties:*

(1) *For all $x \in {}^IW$, there exist objects $Q_x^g \in {}_Q\mathcal{N}^g$, such that*

$$\mathrm{Hom}(Q_x^g, Q_y^g) = \begin{cases} 0 & \text{if } x \neq y; \\ Q_I & \text{if } x = y. \end{cases}$$

Moreover, the objects Q_x^g are precisely the indecomposable objects in ${}_Q\mathcal{N}^g$ up to isomorphism.

(2) *Under the identification $\mathrm{End}(Q_x^g) = Q_I$ of (1), the map*

$$R \to \mathrm{End}(Q_x^g)$$
$$f \mapsto \mathrm{id}_{Q_x^g} \cdot f$$

is given by $f \mapsto x(f) \in Q_I$,

(3) *For all $s \in S$ we have*

$$Q_x^g \cdot B_s = \begin{cases} Q_x^g \oplus Q_{xs}^g & \text{if } xs \in {}^I W \\ 0 & \text{otherwise.} \end{cases}$$

Once one has this theorem the proofs of parts (4) and (5) of Theorem 3.3 are immediate, as in the previous section. (Indeed, one can see the method for the finitary case as explicitly constructing such a module ${}_Q\mathcal{N}^g$ inside the homotopy category of Soergel bimodules.) We now outline how to construct the module ${}_Q\mathcal{N}^g$. (More detail will be provided in a later version of this paper.)

Let us suppose for a moment that I is finitary. Then we know that the indecomposable objects in ${}_Q\mathcal{N}$ are the objects Q_x for $x \in {}^I W$ and that the (obvious analogues of) parts (1), (2) and (3) of the above theorem are satisfied. It is also possible (though tedious) to write down explicit formulas giving the action of each generating morphism (i.e. polynomials, dots, trivalent vertices and $2m_{st}$-valent vertices) on each object Q_x. The result if matrix in Q_I describing a morphism between the result of applying the source (resp. target) of each generating morphism to Q_x. In this way one can describe ${}_Q\mathcal{N}$ as an additive $\mathcal{H}$-module category via generators and relations.

To construct ${}_Q\mathcal{N}^g$ we simply imitate the above formulas for the action. It remains to check that these new formulas define an action of $\mathcal{H}$. To this end, fix a finitary subset $J \subset I$ and an element $x \in {}^I W$ and consider the full additive subcategory of ${}_Q\mathcal{N}^g$ generated by the elements Q_z^g, where z runs over ${}^I W \cap xW_J$. If $x_{\min}$ denotes the minimal element of ${}^I W \cap xW_J$ and J' denotes the set $x_{\min}(J) \cap I$ then one may check that the formulas for the induced action of $\mathcal{H}_J$ on this subcategory agree with the formulas for the J' anti-spherical category of $\mathcal{H}_J$, with the action of polynomials twisted by $x_{\min}$. In particular, any relation satisfied in $\mathcal{H}_J$ is satisfied on this subcategory.

Finally, as the relations for $\mathcal{H}$ all involve finitary subsets $J \subset I$ (even of ranks $1, 2, 3$) we deduce that $\mathcal{H}$ acts on ${}_Q\mathcal{N}^g$ and the theorem follows.

7. I-antispherical double leaves are a basis

In this section we will follow the notation of [EW16, Construction 6.1], where light leaves and double leaves are explained in diagrammatic terms. However we make a slight modification of the construction therein. In the definition of ϕ_k start by doing the following. If $w_{k-1}s \notin {}^I W$ and e_k is either $U0$ or $U1$ then apply some loop in the rex graph starting (and ending) in $\underline{w}_{k-1}s$ and passing through an I-sequence. If not, do nothing. This slight modification in the construction changes nothing in the proof that these morphisms give bases of the corresponding Hom spaces.

Definition 7.1. A subexpression $\mathbf{e}$ of $\underline{x} = s_1 s_2 \cdots s_m$ (we don't consider $\underline{x}$ necessarily reduced) is *I-antispherical* if for all $0 \le k \le m$ we have

$$x_k s_{k+1} \in {}^I W,$$

where x_i is the i^{th}-*point* of the Bruhat stroll. An element of the set $LL_{\underline{x},\mathbf{e}}$ is called an *I-antispherical light leaf* if $\mathbf{e}$ is an I-antispherical subexpression of $\underline{x}$. An *I-antispherical double leaf* is a double leaf where both light leaves that compose it are I-antispherical. Let $\underline{y}$ be another expression. Then we define the set $\mathbb{LL}^{I\text{as}}_{\underline{x},\underline{y}}$ as the subset of $\mathbb{LL}_{\underline{x},\underline{y}}$ consisting of I-antispherical double leaves. We will also denote by $\mathbb{LL}^{I\mathrm{as}}_{\underline{x},\underline{y}}$ this set in $\mathcal{N}$.

Remark 7.2. Let $J \subseteq I \subseteq W$. As ${}^IW \subseteq {}^JW$, we have that if a light leaf is I-antispherical, then it is also J-antispherical.

Theorem 7.3. *If $x, y \in {}^IW$, the set $\mathbb{LL}^{I\mathrm{as}}_{\underline{x},\underline{y}}$ forms a free R_I-basis for $\mathrm{Hom}^{\bullet}{}_{\mathcal{N}}(N_{\underline{x}}, N_{\underline{y}})$.*

Remark 7.4. If x or y are not in ${}^IW$, the theorem still holds, but it says that an empty set is a basis of the zero module.

Proof. In $\mathcal{H}$ the set $\mathbb{LL}_{\underline{x},\underline{y}}$ generates over R (moreover is an R-basis for) the space $\mathrm{Hom}_{\mathcal{H}}(B_{\underline{x}}, B_{\underline{y}})$. By definition of $\mathcal{N}$ we deduce that the set $\mathbb{LL}_{\underline{x},\underline{y}}$ generates over R the space $\mathrm{Hom}_{\mathcal{N}}(N_{\underline{x}}, N_{\underline{y}})$. As $\alpha_s = 0 \in \mathcal{N}$ if $s \in I$, we deduce that the set $\mathbb{LL}_{\underline{x},\underline{y}}$ generates over R_I the space $\mathrm{Hom}_{\mathcal{N}}(N_{\underline{x}}, N_{\underline{y}})$.

But it is easy to see that with the slightly modified construction of light leaves a light leaf that is not I-antispherical is zero in $\mathcal{N}$. Indeed, there is some $1 \leq k \leq m$ such that $w_k s_{k+1} \notin {}^IW$. Consider the lowest k with that property. If $k = 1$ then this means that $w_1 = 1$. If $k > 1$ then $w_{k-1}s_k \in {}^IW$, thus in any case $w_k \in {}^IW$. By Fact (c) in § 2.5 we have the inequality $w_k s_{k+1} > w_k$. So we have that e_{k+1} is either $U0$ or $U1$, so by the slight modification of the construction, the light leaf factors through some I-sequence, thus is zero. So we have proved that the I-antispherical double leaves generate the space $\mathrm{Hom}_{\mathcal{N}}(N_{\underline{x}}, N_{\underline{y}})$ over R_I.

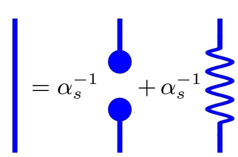
We just need to see that the set $\mathbb{LL}^{I\mathrm{as}}_{\underline{x},\underline{y}}$ is free over R_I. The proof is the standard localization proof that double leaves are free ([EW16, Proposition 6.9], [Lib15]). We will prove that this basis has an upper-triangularity property for the double path dominance order, defined earlier. Let us be more precise.

Lemma 7.5. *For any sequence $\underline{x}$ of elements in S we have the following decomposition in ${}_Q\mathcal{N}$*

$$N_{\underline{x}} = \bigoplus_{\substack{\mathbf{e} \subseteq \underline{x} \\ I-\text{antispherical}}} Q_{\mathbf{e}}$$

Proof. This is a direct consequence of part 3 of Theorem 3.3. ☐

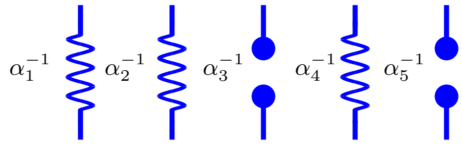
The inclusion $\mathrm{inc}_{\mathbf{e}} : Q_{\mathbf{e}} \to N_{\underline{x}}$ and the projection $\mathrm{pro}_{\mathbf{e}} : N_{\underline{x}} \to Q_{\mathbf{e}}$ are given by the same endomorphism of $N_{\underline{x}}$, the one obtained by the equality in ${}_Q\mathcal{N}$



Let us give an example, (when $y(\alpha_s) \notin \Phi_I$ we will denote by $Q_y\alpha_s^{-1}$ the morphism $y(\alpha_s)^{-1}Q_y$ in ${}_Q\mathcal{N}$). The inclusion

$$\mathrm{inc}_{(1,1,0,1,0)} : Q_{(1,1,0,1,0)} \to N_{s_1s_2s_3s_4s_5}$$

is given by the following picture (where $\alpha_i := \alpha_{s_i}$)



Recall that if $LL_{\underline{x},\mathbf{e}}\colon B_{\underline{x}} \to B_{\underline{w}}$ is a light leaves map where $\underline{w}$ is a rex for w, by flipping this diagram upside-down, we get a map $\overline{LL}_{\underline{x},\mathbf{e}}\colon B_{\underline{w}} \to B_{\underline{x}}$.

Let $\underline{x}$ and $\underline{y}$ be arbitrary sequences with subsequences $\mathbf{e}$ and $\mathbf{f}$ respectively, such that $(\underline{x},\mathbf{e})$ and $(\underline{y},\mathbf{f})$ both express w. Choose a rex $\underline{w}$ for w, and construct maps $LL_{\underline{x},\mathbf{e}}\colon B_{\underline{x}} \to B_{\underline{w}}$ and $\overline{LL}_{\underline{y},\mathbf{f}}\colon B_{\underline{w}} \to B_{\underline{y}}$. The corresponding double leaves map is the composition

$$\mathbb{LL}_{w,\mathbf{f},\mathbf{e}} \stackrel{\text{def}}{=} \overline{LL}_{\underline{y},\mathbf{f}} \circ LL_{\underline{x},\mathbf{e}}.$$

In ${}_Q\mathcal{N}$, this morphism gives a coefficient $p^{\mathbf{f},\mathbf{e}}_{\mathbf{f}',\mathbf{e}'} \in Q_I$ given by the inclusion of each standard summand $Q_{\mathbf{e}'}$ of $N_{\underline{x}}$ and projection to each standard summand $Q_{\mathbf{f}'}$ of $N_{\underline{y}}$, in the sense of Lemma 7.5. The following facts about these coefficients are easy:

- $p^{\mathbf{f},\mathbf{e}}_{\mathbf{f}',\mathbf{e}'} = 0$ unless $(\underline{x},\mathbf{e}')$ and $(\underline{y},\mathbf{f}')$ express the same element v. This is a direct consequence of part 4 of Theorem 3.3.
- $p^{\mathbf{f},\mathbf{e}}_{\mathbf{f}',\mathbf{e}'} = 0$ unless both $\mathbf{e}' \leq \mathbf{e}$ and $\mathbf{f}' \leq \mathbf{f}$. This is a direct consequence of the construction of light leaves, Equation 4.3 and again part 4 of Theorem 3.3.
- The element $p^{\mathbf{f},\mathbf{e}}_{\mathbf{f},\mathbf{e}}$ is invertible in Q_I. Moreover, it is a product of roots, obeying a simple formula independent of the choice of LL maps (for more details see [EW16, Proposition 6.6]).

Consider the double path dominance order introduced in § 3.5, restricted to pairs of 01-sequences with the same fixed end-point. As we have seen, $\mathbb{LL}$ maps satisfy upper-triangularity with respect to this partial order, with an invertible diagonal, thus giving linear independence of $\mathbb{LL}^{I\text{as}}_{\underline{x},\underline{y}}$ over R_I. □

8. Categorification theorem

Recall that D_x denotes the image of the indecomposable object B_x in $\mathcal{N}$. By the definition of $\mathcal{N}$ as a quotient of additive categories, it is clear that the set

$$\{D_x \mid x \in {}^IW\}$$

is a set of representatives for the isomorphism classes of indecomposable objects of $\mathcal{N}$, up to shift. (They are non-zero because their images in ${}_Q\mathcal{N}$ are non-zero, by Remark 6.2.) For any $x \in {}^IW$ consider the full additive subcategory

$$\mathcal{N}_{\not\geq x} := \langle D_y(m) \mid y \not\geq x \text{ and } m \in \mathbb{Z}\rangle_{\oplus}$$

and the quotient (of additive categories)

$$\mathcal{N}^{\geq x} := \mathcal{N}/\mathcal{N}_{\not\geq x}.$$

Lemma 8.1. *For any expression* $\underline{y}$*,* $\operatorname{Hom}^\bullet_{\mathcal{N}^{\geq x}}(N_{\underline{y}}, D_x)$ *is a free graded* R_I*-module, with basis indexed by the (images of) the* I*-antispherical light leaves corresponding to* I*-antispherical subexpressions of* $\underline{y}$ *for* x*.*

Proof. Let $\underline{x}$ be a reduced expression for x. By Theorem 7.3, $\mathrm{Hom}^\bullet{}_{\mathcal{N}^{\ge x}}(N_{\underline{y}}, N_{\underline{x}})$ is free over R_I with basis given by I-antispherical double leaves. However when we pass to the quotient $\mathcal{N}^{\ge x}$ all double leaves with non-trivial upper light leaf, factor through an object in $\mathcal{N}_{\not\ge x}$ and are therefore zero. We conclude that the claimed elements span $\mathrm{Hom}^\bullet{}_{\mathcal{N}^{\ge x}}(N_{\underline{y}}, D_x)$.

To see that they are linearly independent, consider the chain of functors

$$\mathcal{N} \to {}_Q\mathcal{N} \to {}_Q\mathcal{N}/\langle Q_y \mid y \not\ge x\rangle_\oplus$$

where the first functor is given by localisation, and the second is the quotient functor. If $y \not\ge x$, the image of D_y is zero, and hence we obtain a functor

$$\mathcal{N}^{\ge x} \to {}_Q\mathcal{N}/\langle Q_y \mid y \not\ge x\rangle_\oplus$$

By Theorem 7.3 the images of the light leaves as in the lemma are linearly independent on the right hand side, and hence are already linearly independent on the left hand side. Thus the statement of the lemma is true for $\mathrm{Hom}^\bullet{}_{\mathcal{N}^{\ge x}}(N_{\underline{y}}, N_{\underline{x}})$. Finally, D_x and $N_{\underline{x}}$ are isomorphic in $\mathcal{N}^{\ge x}$ and the lemma follows. □

Because any object in $\mathcal{N}$ is a direct sum of shifts of summands of $N_{\underline{x}}$ we conclude that the space $\mathrm{Hom}^\bullet{}_{\mathcal{N}^{\ge x}}(M, D_x)$ is a free graded R_I-module for any object $M \in \mathcal{N}$. We define the *diagrammatic character* as follows

$$\begin{aligned} \mathrm{ch} : [\mathcal{N}] &\to N \\ [M] &\mapsto \sum_{x \in {}^I W} \mathrm{grk}\,\mathrm{Hom}^\bullet{}_{\mathcal{N}^{\ge x}}(M, D_x) n_x, \end{aligned}$$

where grk denotes graded rank.

Theorem 8.2. *The diagrammatic character gives an isomorphism*

$$\mathrm{ch} : [\mathcal{N}] \xrightarrow{\sim} N$$

as $[\mathcal{H}] = H$*-modules. Under this isomorphism the indecomposable object* D_x *is mapped to the Kazhdan-Lusztig basis* d_x*.*

Proof. It is clear that ch is a morphism of $\mathbb{Z}[v^{\pm 1}]$-modules. As explained above, the set $\{D_x | x \in {}^I W\}$ give representatives for the indecomposable objects of $\mathcal{N}$ up to shifts and isomorphism. Hence

$$[\mathcal{N}] = \bigoplus \mathbb{Z}[v^{\pm 1}][D_x].$$

On the other hand, it is immediate from the definition that

$$\mathrm{ch}([D_x]) = n_x + \sum_{y < x} n'_{y,x} n_y$$

for some $n'_{y,x} \in \mathbb{Z}_{\ge 0}[v^{\pm 1}]$. We conclude that ch maps a basis of $[\mathcal{N}]$ to a basis of N, and hence is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-modules.

For any expression $\underline{y} = s_1 \dots s_m$ set $d_{\underline{y}} := n_{\mathrm{id}} \cdot b_{s_1} \dots b_{s_m}$. Lemma 8.1 combined with (2.5) implies by induction on m that $\mathrm{ch}([N_{\underline{y}}]) = d_{\underline{y}}$. For any $s \in S$ we have tautologically

$$\mathrm{ch}([N_{\underline{y}}][B_s]) = \mathrm{ch}([N_{\underline{y}'}]) = d_{\underline{y}'} = d_{\underline{y}} \cdot b_s$$

where $\underline{y}' = s_1 \dots s_m s$. We conclude that ch is a map of $[\mathcal{H}] = H$-modules on the $\mathbb{Z}[v^{\pm 1}]$-submodule generated by $[N_{\underline{y}}]$, where $\underline{y}$ ranges over all expressions. However this submodule is all of $[\mathcal{N}]$ and hence ch is an isomorphism of $[\mathcal{H}] = H$-modules.

We will prove by induction in $l(x)$ that $\mathrm{ch}([D_x]) = d_x$, so let us suppose that we know this equality for all y such that $l(y) < l(x)$. Let $\underline{x}$ be a reduced expression for $x \in {}^I W$. Then in $\mathcal{H}$ we have

$$B_{\underline{x}} = B_x + E$$

where E is some self-dual object, all of whose indecomposable summands are parametrized by $y < x$. By acting on N_{id} we conclude that

$$N_{\underline{x}} = D_x + \overline{E}$$

where $\overline{E}$ is a self-dual combination of D_y with $y < x$ and $y \in {}^I W$. As observed above, we have $\mathrm{ch}([N_{\underline{x}}]) = d_{\underline{x}}$ and so is self-dual. By induction, $\mathrm{ch}([\overline{E}])$ is self-dual. We deduce that $\mathrm{ch}([D_x])$ is self-dual as well, as the difference of two self-dual elements.

Finally, by the main theorem of [EW14] (more precisely, see second sentence following [EW14, Theorem 2.6]) we know that $\mathrm{Hom}^{\bullet}{}_{\mathcal{N}^{\geq y}}(D_x, D_y)$ is generated in strictly positive degrees for $y < x$. We conclude that the polynomials $n'_{y,x}$ defined above actually satisfy $n'_{y,x} \in v\mathbb{Z}[v]$ for $y < x$. Hence by the uniqueness of the Kazhdan-Lusztig basis we deduce that

$$\mathrm{ch}([D_x]) = d_x.$$

The theorem follows. □

Corollary 8.3. *The anti-spherical Kazhdan-Lusztig polynomials $n_{y,x}$ have non-negative coefficients.*

Also, if $J \subset I \subset S$ it is immediate (either from Remark 7.2 or from the fact that $\mathcal{N}_I$ is a quotient of $\mathcal{N}_J$) that we have a surjection

$$\mathrm{Hom}^{\bullet}{}_{\mathcal{N}_J^{\geq y}}(D_x^J, D_y^J) \twoheadrightarrow \mathrm{Hom}^{\bullet}{}_{\mathcal{N}_I^{\geq y}}(D_x^I, D_y^I)$$

and we deduce:

Corollary 8.4. *Brenti's Monotonicity conjecture: $J \subseteq I$ implies that $n^I_{y,x} \leq n^J_{y,x}$.*

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