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New bases of some Hecke algebras via Soergel bimodules

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Abstract

For extra-large Coxeter systems ($m(s,r) > 3$), we construct a natural and explicit set of Soergel bimodules $D = \{D_w\}_{w \in W}$ such that each D_w contains as a direct summand (or is equal to) the indecomposable Soergel bimodule B_w. When decategorified, we prove that D gives rise to a set $\{d_w\}_{w \in W}$ that is actually a basis of the Hecke algebra. This basis is close to the Kazhdan–Lusztig basis and satisfies a positivity condition.

Keywords: Soergel bimodules; Kazhdan–Lusztig polynomials; Hecke algebra

1. Introduction

The understanding of indecomposable Soergel bimodules has close relations with combinatorics, representation theory and knot theory.

Indeed, indecomposable Soergel bimodules provide an approach to proving the positivity of Kazhdan–Lusztig polynomials via Soergel's conjecture, which gives a conjectural interpretation of the coefficients.

This research direction also gives the main algebraic/combinatorial approach to finding character formulas for simple modules for algebraic groups. Soergel bimodules provide a framework for attacking part of the Lusztig conjecture [10] in characteristic bigger than the Coxeter number. They may also provide a means to conjecture what happens in small characteristic.

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These indecomposable bimodules also play an important role in one approach to the calculation of Khovanov–Rozansky homology of links [6].

One might see this paper as a first step in the program of explicitly constructing these elusive indecomposable bimodules. We expect that with a refinement of our methods one might find all the indecomposable bimodules for large Coxeter groups (*i.e.* all $m(s,r) \neq 2$), thus approaching to a solution of the Kazhdan–Lusztig positivity conjecture in these cases. This might also give an idea of how to find the indecomposable bimodules for a general Coxeter system.

Let us be more precise. Consider a Coxeter system $(W, \mathcal{S})$, and its geometric representation V. Let $R = R(V)$ be the algebra of regular functions on V. We equip R with a $\mathbb{Z}$-grading $R = \bigoplus_{i \in \mathbb{Z}} R_i$ given by $R_2 = V^*$ and $R_i = 0$ for uneven i. The action of W on V induces an action on R. For $s \in \mathcal{S}$ consider the (R, R)-bimodule $\theta_s = R \otimes_{R^s} R$, where R^s is the subspace of R fixed by s.

The category of *Soergel bimodules* is the category with objects the $\mathbb{Z}$-graded (R, R)-bimodules obtained as shifts of finite direct sums of direct summands of bimodules of the type $\theta_{s_1} \otimes_R \theta_{s_2} \otimes_R \cdots \otimes_R \theta_{s_n}(d) := \theta_{s_1}\theta_{s_2}\cdots\theta_{s_n}$ for $(s_1, \ldots, s_n) \in \mathcal{S}^n$.

There exists an isomorphism of rings $\varepsilon : \mathcal{H} \to \langle \mathbf{B} \rangle$, where $\mathcal{H}$ is the Hecke algebra of $(W, \mathcal{S})$ and $\langle \mathbf{B} \rangle$ is the split Grothendieck group of $\mathbf{B}$. This map may be used to view $\mathbf{B}$ as a categorification of the Hecke algebra.

The indecomposable objects B_x of $\mathbf{B}$ are parametrized (up to isomorphism and shifts in the grading) by elements $x \in W$ of the Coxeter group. If $\bar{s} = (s_1, \ldots, s_n)$ is a reduced expression of $x \in W$, then B_x is a direct summand of the bimodule $\theta_{\bar{s}} = \theta_{s_1}\theta_{s_2}\cdots\theta_{s_n}$.

For $s, r \in \mathcal{S}$, we have introduced in the paper [7] a morphism f_{sr} satisfying that it is the unique (up to a scalar) degree zero morphism in the space $\mathrm{Hom}_{\mathbf{B}}(\underbrace{\theta_s\theta_r\theta_s\cdots}_{m(s,r)}, \underbrace{\theta_r\theta_s\theta_r\cdots}_{m(s,r)})$. At this point is natural to ask what happens if we apply "all possible" morphisms of the form $\mathrm{id} \otimes f_{sr} \otimes \mathrm{id}$ to $\theta_{\bar{s}}$? In fact, if $(W, \mathcal{S})$ is an extra-large Coxeter system, by applying all such morphisms one obtains an idempotent projector and its image is a well-defined Soergel bimodule. Note that a priori it is not clear that one obtains in this way an idempotent.

More precisely we consider the graph $\mathcal{R}(x)$ whose vertices consist of reduced expressions for x, and edges corresponding to braids relations. To each circuit c in $\mathcal{R}(x)$ with starting point $\bar{s}$ which visits every vertex we associate a natural idempotent $f^e_{\bar{s},c}$ given by composing morphisms of the form $\mathrm{id} \otimes f_{sr} \otimes \mathrm{id}$ along c. We prove that the image $E_x = \mathrm{Im}(f^e_{\bar{s},c})$ does not depend (up to isomorphism) on the choice of the reduced expression $\bar{s}$, or on the choice of the path c.

Although B_x is a direct summand of E_x, these two bimodules will rarely be isomorphic. The problem is that we have to examine more morphisms than those of the form f_{sr}. Let us define, for $s, r \in \mathcal{S}$ and $1 \leqslant n \leqslant m(s,r)$ the element $sr(n) = \underbrace{srs\cdots}_{n}$. We can describe $f_{rs} \circ f_{sr}$ as the unique degree zero idempotent that factors through the indecomposable $B_{sr(m)}$, with $m = m(s,r)$. So, by analogy we define $f^2_{sr}(n)$ as the unique idempotent in $\mathrm{End}(\underbrace{\theta_s\theta_r\theta_s\cdots}_{n})$ that factors through $B_{sr(n)}$. We remark that $f^2_{sr}(n)$ is not defined over all fields because there are denominators involved in its definition, and this is consistent with the fact that Soergel's conjecture is not expected to be true in all characteristics.

It is again natural to ask what happens if we apply "all possible" morphisms of the form $\mathrm{id} \otimes f_{sr} \otimes \mathrm{id}$ and morphisms of the form $\mathrm{id} \otimes f^2_{sr}(n) \otimes \mathrm{id}$ with $n > 3$ to $\theta_{\bar{s}}$?

The answer is the same as before. In the extra-large case we obtain an idempotent $f^d_{\bar{s},c}$ of $\mathrm{End}(\theta_{\bar{s}})$ whose image is $D_x = \mathrm{Im}(f^d_{\bar{s},c})$, a bimodule depending only on x.

We prove that B_x is a direct summand of D_x, which is itself a direct summand of E_x. So we have a chain $B_x \subseteq D_x \subseteq E_x \in \mathbf{B}$ of bimodules in Soergel category. We prove that the two sets $\{\eta(\langle D_x\rangle)\}_{x\in W}$ and $\{\eta(\langle E_x\rangle)\}_{x\in W}$ are bases of the Hecke algebra, where $\eta : \langle \mathbf{B}\rangle \to \mathcal{H}$ is by definition the inverse morphism of ε. The problem of giving an explicit formula for $\eta(\langle D_x\rangle)$ or for $\eta(\langle E_x\rangle)$ seems quite difficult. We don't even have a conjecture yet.

By general arguments due to Soergel, when the elements of these bases are written in the form $\sum h_w T_w$, the h_w have positive coefficients. The Kazhdan–Lusztig basis (conjecturally corresponding to the indecomposable Soergel bimodules) is expected to have the same positivity property. We expect that all the results just cited will generalize to Weyl groups.

The paper is organized as follows. In Section 2 we define Soergel bimodules and discuss Soergel's conjecture. In Section 3 we give a way to calculate the morphism f_{sr} via symmetric algebras. In Section 4 we prove a theorem that is the technical heart of this paper. In Section 5 we define E_x and discuss its properties. Finally in Section 6 we define D_x and discuss its properties.

2. Soergel's conjecture

2.1. *Soergel's category* B

Let $(W, \mathcal{S})$ be a not necessarily finite Coxeter system (with $\mathcal{S}$ a finite set) and $\mathcal{T} \subset W$ the set of reflections in W, *i.e.* the orbit of $\mathcal{S}$ under conjugation. Let k be an infinite field of characteristic different from 2 and V a finite dimensional k-representation of W. For $w \in W$, we denote by $V^w \subset V$ the set of w-fixed points. The following definition can be found in [12].

Definition 2.2. By a *reflection faithful representation* of $(W, \mathcal{S})$ we mean a faithful, finite dimensional representation V of W such that for each $w \in W$, the subspace V^w is a hyperplane of V if and only if $w \in \mathcal{T}$.

From now on, we consider V a reflection faithful representation of W. If $k = \mathbb{R}$, by the results of [8], all the results in this paper will stay true if we consider V to be the geometric representation of W (even if this representation in not always reflection faithful). So if the reader is not interested in positive characteristic or if they are only interested in the positivity of the coefficients of Kazhdan–Lusztig polynomials they can suppose from now on that $k = \mathbb{R}$ and that V is the geometric representation of W.

Let $R = R(V)$ be the algebra of regular functions on V. The algebra R has the following grading: $R = \bigoplus_{i\in\mathbb{Z}} R_i$ with $R_2 = V^*$ and $R_i = 0$ if i is odd. The action of W on V induces an action on R. For $s \in \mathcal{S}$ consider the (R, R)-bimodule $\theta_s = R \otimes_{R^s} R$, where R^s is the subspace of R fixed by s.

We denote by $\mathcal{R}$ the category of all $\mathbb{Z}$-graded R-bimodules, which are finitely generated from the right as well as from the left, and where the action of k from the right and from the left is the same. For every graded object $M = \bigoplus_i M_i$, and every integer n, we denote by $M(n)$ the shifted object defined by the formula $(M(n))_i = M_{i+n}$. Now we can define Soergel's bimodule category.

Definition 2.3. *Soergel's category* $\mathbf{B}(W, V) = \mathbf{B}$ is the full subcategory of $\mathcal{R}$ with objects the finite direct sums of direct summands of bimodules of the type $\theta_{s_1} \otimes_R \theta_{s_2} \otimes_R \cdots \otimes_R \theta_{s_n}(d)$ for $(s_1, \ldots, s_n) \in \mathcal{S}^n$, and $d \in \mathbb{Z}$.

By convention, we will denote by $\theta_{s_1}\theta_{s_2}\cdots\theta_{s_n}$ the (R, R)-bimodule

$$\theta_{s_1} \otimes_R \theta_{s_2} \otimes_R \cdots \otimes_R \theta_{s_n} \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R.$$

2.4. Soergel's categorification and conjecture

Before we can state Soergel's categorification of the Hecke algebra we will recall what the Hecke algebra and the split Grothendieck group are.

Definition 2.5. Let $(W, \mathcal{S})$ be a Coxeter system. The *Hecke algebra* $\mathcal{H} = \mathcal{H}(W, \mathcal{S})$ is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $\{T_s\}_{s\in\mathcal{S}}$, and relations

$$T_s^2 = v^{-2} + \big(v^{-2} - 1\big)T_s \quad \text{for all } s \in \mathcal{S} \quad \text{and}$$

$$\underbrace{T_s T_r T_s \cdots}_{m(s,r)\text{ terms}} = \underbrace{T_r T_s T_r \cdots}_{m(s,r)\text{ terms}} \quad \text{if } s, r \in \mathcal{S} \text{ and } sr \text{ is of order } m(s, r).$$

If $x = s_1 s_2 \cdots s_n$ is a reduced expression of x, we define $T_x = T_{s_1}T_{s_2}\cdots T_{s_n}$ (T_x does not depend on the choice of the reduced expression). We put $q = v^{-2}$.

Definition 2.6. For every essentially small abelian category $\mathcal{A}$, we define the *split Grothendieck group* $\langle\mathcal{A}\rangle$. It is the free abelian group generated by the objects of $\mathcal{A}$ modulo the relations $M = M' + M''$ whenever we have $M \cong M' \oplus M''$. Given an object $A \in \mathcal{A}$, let $\langle A\rangle$ denote its class in $\langle\mathcal{A}\rangle$.

The following theorem can be found in [12].

Theorem 2.7. *Let $(W, \mathcal{S})$ be a Coxeter system and $\mathcal{H}$ its Hecke algebra. There exists a unique ring isomorphism $\varepsilon : \mathcal{H} \to \langle\mathbf{B}\rangle$ such that $\varepsilon(v) = \langle R(1)\rangle$ and $\varepsilon(T_s + 1) = \langle\theta_s\rangle$ for all $s \in \mathcal{S}$.*

The following theorem can be found in [5].

Theorem 2.8. *Let us define in the Hecke algebra the elements $\tilde{T}_x = v^{l(x)}T_x$. There exists a unique involution $d : \mathcal{H} \to \mathcal{H}$ with $d(v) = v^{-1}$ and $d(T_x) = (T_{x^{-1}})^{-1}$. For $x \in W$ there exists a unique $C'_x \in \mathcal{H}$ with $d(C'_x) = C'_x$ and*

$$C'_x \in \tilde{T}_x + \sum_y v\mathbb{Z}[v]\tilde{T}_y.$$

These elements form the so-called Kazhdan–Lusztig basis of the Hecke algebra.

Now we can state Soergel's conjecture.

Conjecture 2.9 *(Soergel). For every $x \in W$, there exists an indecomposable $\mathbb{Z}$-graded R-bimodule $B_x \in \mathcal{R}$ such that $\varepsilon(C'_x) = \langle B_x\rangle$.*

Remark 2.10. In [12] Soergel proves that this conjecture implies the positivity of all coefficients of all Kazhdan–Lusztig polynomials. By the work of Soergel [11] and Fiebig [3] we know that for W a finite Weyl group and char(k) at least the Coxeter number, this conjecture is equivalent to Lusztig's conjecture concerning characters of irreducible representations of algebraic groups over k.

3. A categorification of the braid relation

3.1. The morphism f_{sr}

We start this section with some notation. Let $s, r \in \mathcal{S}$, $s \neq r$, with $m(s,r) \neq \infty$. We denote by X_{sr} the bimodule $\underbrace{\theta_s\theta_r\theta_s\cdots}_{m(s,r)\text{ terms}}$ and we denote by DZ_{sr} the space of degree zero morphisms of (R, R)-bimodules from X_{sr} to X_{rs}.

The following proposition can be found in [7, Proposition 4.3].

Proposition 3.2. *Let $s \neq r \in \mathcal{S}$ with $m(s,r) \neq \infty$. The space DZ_{sr} is one-dimensional.*

Recall that $\mathcal{T} \subseteq W$ is the subset of reflections of W. For each $t \in \mathcal{T}$, let Y_t be the subset of V^* of linear forms with kernel equal to the hyperplane fixed by t. It is clear that if $y, y' \in Y_t$, then there exists $0 \neq \lambda \in k$ such that $y = \lambda y'$. Let $s, r \in \mathcal{S}$. If we choose one element $x_s \in Y_s$ and one element $x_r \in Y_r$, and we put

$$t = \begin{cases} s & \text{if } m \text{ is odd,} \\ r & \text{if } m \text{ is even,} \end{cases} \qquad u = \begin{cases} r & \text{if } m \text{ is odd,} \\ s & \text{if } m \text{ is even} \end{cases}$$

by [7, Lemme 4.7] there exists a unique element $f_{sr} \in DZ_{sr}$ with

$$f_{sr}(1 \otimes x_s \otimes x_r \otimes x_s \otimes \cdots \otimes x_t) \in 1 \otimes x_r \otimes x_s \otimes x_r \otimes \cdots \otimes x_u + R_+X_{rs},$$

where R_+ is the ideal of R generated by the homogeneous elements of non-zero degree. We insist on the fact that f_{sr} depends on the choice of x_s and of x_r.

3.3. A formula for f_{sr}

In this section we find a formula for f_{sr}, using the fact that R is a symmetric algebra over $R^{\langle s,r\rangle}$. We fix a dihedral Coxeter system $(W_0, \mathcal{S}_0)$ with $\mathcal{S}_0 = \{s, r\}$ and $m = m(s,r)$. We will suppose that the order of W_0 is invertible in the field k.

In this section we will fix, for every reflection $t \in \mathcal{T}_0 \subset W_0$, an element $x_t \in Y_t$. For all $s' \in \mathcal{S}_0$ we define $\partial_{s'} : R \to R$ by the formula $\partial_{s'}(a) = (a - s' \cdot a)/2x_{s'}$. We recall the following theorem (see [2, Théorème 2]).

Theorem 3.4.

(1) *If $w \in W_0$ and $(s_1, \ldots, s_n)$ is a reduced expression of w then the element $\partial_w = \partial_{s_1} \cdots \partial_{s_n}$ depends only on w; it does not depend on the choice of the reduced expression.*
(2) *If $d = \prod_{t \in \mathcal{T}_0} x_t$, then the set $\{\partial_w(d)\}_{w \in W_0}$ is a basis of R as R^{W_0}-module.*

So we can define the graded R^{W_0}-module morphism

$$\hat{t} : R \to R^{W_0}(-2m)$$
$$\sum_{w \in W_0} \lambda_w \partial_w(d) \mapsto \lambda_1.$$

The following lemma is classical.

Lemma 3.5. *R is a symmetric algebra over R^{W_0} and $\hat{t}$ is the symmetrizing form.*

Proof. By definition $\hat{t}$ is a linear form and as R is commutative, it is trivial to see that $\hat{t}$ is a trace. We only need to prove that the map $\psi : R(2m) \to \mathrm{Hom}_{R^{W_0}}(R, R^{W_0})$ that sends a to $(b \to \hat{t}(ab))$ is a graded isomorphism of R-modules. The fact that ψ is a morphism of R-modules is a consequence of the fact that R is commutative.

To see that ψ is injective, we use the easily verifiable fact that $\hat{t}(\partial_x(d)\partial_y(d)) \in \delta_{xw_0,y} + R_+$. Finally, using the isomorphism

$$R \cong \bigoplus_{w \in W_0} R^{W_0}\big(-2l(w)\big) \text{ as left graded } R^{W_0}\text{-mod} \tag{3.1}$$

we conclude easily that there exists an isomorphism of graded R^{W_0}-modules $\mathrm{Hom}_{R^{W_0}}(R, R^{W_0}) \cong R(2m)$, and so ψ is an isomorphism. □

Let us consider the dual basis $\{\partial_w(d)^*\}_{w \in W_0}$ with respect to the linear form $\hat{t}$:

$$\hat{t}\big(\partial_w(d)\partial_{w'}(d)^*\big) = \delta_{w,w'}.$$

The objects of this dual basis are homogeneous. By [1, Lemma 3.2], we have that $\sum_{w \in W_0} \partial_w(d) \otimes \partial_w(d)^*$ is the Casimir element and it is clear that $\deg(\partial_w(d) + \partial_w(d)^*) = 2m$, so, by [1, Proposition 3.3], the map

$$\xi : R \to R \otimes_{R^{W_0}} R(2m)$$
$$1 \mapsto \sum_{w \in W_0} \partial_w(d) \otimes \partial_w(d)^*$$

is a non-zero morphism of graded (R, R)-bimodules.

Let us consider the decomposition $R = R^s \oplus x_s R^s$. Let $p_1, p_2 \in R^s$ and $p, q, r \in R$. We define the morphisms of graded R^s-modules:

$$P_s : R \to R, \quad p_1 + x_s p_2 \mapsto p_1,$$
$$I_s : R \to R, \quad p_1 + x_s p_2 \mapsto x_s p_2,$$
$$\partial_s : R(2) \to R, \quad p_1 + x_s p_2 \mapsto p_2,$$

and the morphisms of graded (R, R)-bimodules:

$$m_s : \theta_s \to R, \quad R \otimes_{R^s} R \ni p \otimes q \mapsto pq,$$
$$j_s : \theta_s \theta_s(2) \to \theta_s, \quad R \otimes_{R^s} R \otimes_{R^s} R \ni p \otimes q \otimes r \mapsto p\partial_s(q) \otimes r \in R \otimes_{R^s} R.$$

Let

$$t = \begin{cases} s & \text{if } m \text{ is odd,} \\ r & \text{if } m \text{ is even,} \end{cases} \qquad u = \begin{cases} r & \text{if } m \text{ is odd,} \\ s & \text{if } m \text{ is even.} \end{cases}$$

We can now define two morphisms of graded (R, R)-bimodules:

$$\iota : R \otimes_{R^{W_0}} R(-2m) \to R \otimes_{R^t} R \otimes \cdots \otimes_{R^s} R \otimes_{R^{W_0}} R(-2m) \simeq X_{tu} \otimes_{R^{W_0}} R(-2m)$$
$$a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes b$$

and $\vartheta : X_{sr} \otimes_R X_{tu} \to R(-2m)$ defined by

$$\vartheta = (m_s \circ j_s) \circ \big(\mathrm{id}^1 \otimes (m_r \circ j_r) \otimes \mathrm{id}^1\big) \circ \big(\mathrm{id}^2 \otimes (m_s \circ j_s) \otimes \mathrm{id}^2\big) \circ \cdots$$
$$\circ \big(\mathrm{id}^{m-1} \otimes (m_t \circ j_t) \otimes \mathrm{id}^{m-1}\big).$$

We introduce the morphism $\Phi \in \mathrm{Hom}(X_{sr}, R \otimes_{R^{W_0}} R)$, defined by

$$\Phi = (\vartheta \otimes \mathrm{id}_{R \otimes_{R^{W_0}} R}) \circ \big(\mathrm{id}_{X_{sr}} \otimes (\iota \circ \xi)\big),$$

where we identify the domain X_{sr} with $X_{sr} \otimes_R R$.

Finally we can define the following graded morphism

$$\Psi : R \otimes_{R^{W_0}} R \to X_{rs}$$
$$a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes b.$$

Proposition 3.6. *The morphism $\Psi \circ \Phi \in \mathrm{Hom}(X_{sr}, X_{rs})$ is a non-zero scalar multiple of f_{sr}. If we identify $X_{sr} = R \otimes_{R^s} R \otimes_{R^r} R \otimes_{R^s} R \otimes \cdots \otimes_{R^t} R$ and $X_{rs} = R \otimes_{R^r} R \otimes_{R^s} R \otimes_{R^r} R \otimes \cdots \otimes_{R^u} R$ we explicitly have*

$$\Psi \circ \Phi(p_0 \otimes p_1 \otimes \cdots \otimes p_n) = \sum_{w \in W_0} p_0 \partial_s\big(p_1 \partial_r\big(p_2 \partial_s\big(p_3 \cdots \partial_t\big(p_n \partial_w(d)\big) \cdots\big)\big)\big)$$
$$\otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \partial_w(d)^*.$$

Proof. It is clear that $\Psi \circ \Phi$ is a degree zero morphism, then by definition of f_{sr} we only need to prove that $\Psi \circ \Phi \neq 0$. But for this we only need to note that $\Psi \circ \Phi(1 \otimes 1 \otimes \cdots \otimes 1) = \partial_{w_0}(d) \otimes 1 \otimes \cdots \otimes 1$, where w_0 is the longest element of W_0. As $\partial_{w_0}(d)$ is part of the basis in Theorem 3.4, it is non-zero. □

Remark 3.7. If $\Psi \circ \Phi$ is a scalar multiple of f_{sr}, we can choose some $t_0 \in \mathcal{T}_0$ with $t_0 \notin \mathcal{S}_0$ and change the definition of x_{t_0} (by a scalar multiple) so as to change by a scalar multiple the definition of $d = \prod_{t \in \mathcal{T}_0} x_t$ and thus to have exactly $\Psi \circ \Phi = f_{sr}$.

Remark 3.8. Let $(W, \mathcal{S})$ be any Coxeter system (not necessarily dihedral), V a reflection faithful k-representation of W and $s', r' \in \mathcal{S}$. The action of the subgroup $\langle s', r'\rangle$ of W gives a decomposition $V = V_1 \oplus V_2$, with $V_1 = V^{\langle s', r'\rangle}$ and V_2 the geometric representation of the dihedral Coxeter system $(\langle s', r'\rangle, \{s', r'\})$. We have that $R(V) = R(V_1) \otimes_k R(V_2)$ and the functor $- \otimes_{R(V)^{s'}} R(V)$ is isomorphic to the functor $- \otimes_{R(V_2)^{s'}} R(V_2)$ (the same isomorphism is true if we replace s' by r'). So we can obtain Proposition 3.6 for V from Proposition 3.6 for V_2 by applying the functor $R(V_1) \otimes_k -$. In other words, we obtain in such a way a formula for $f_{s'r'}$ for any Coxeter system when $m(s', r')$ is invertible in the field k.

3.9. An important property of f_{sr}

In this section we prove a property of f_{sr} that will be useful in Section 4. We start with a trivial corollary of Proposition 3.6.

Corollary 3.10. $f_{sr}(1 \otimes_{R^s} \theta_r\theta_s \cdots \theta_t) \subseteq R^s \otimes_{R^r} 1 \otimes_{R^s} 1 \otimes \cdots 1 \otimes_{R^u} R$.

We need some definitions in order to state the next proposition.

Definition 3.11. When $i \in \mathbb{N}$ is such that

$$\mathrm{id}^i \otimes f_{s_{i+1}s_{i+2}} \otimes \mathrm{id}^{n-i-m(s_{i+1},s_{i+2})} \in \mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, \theta_{t_1} \cdots \theta_{t_n}),$$

then we denote by ${}^i f$ the morphism $\mathrm{id}^i \otimes f_{s_{i+1}s_{i+2}} \otimes \mathrm{id}^{n-i-m(s_{i+1},s_{i+2})}$. A morphism g between $\theta_{s_1} \cdots \theta_{s_n}$ and $\theta_{t_1} \cdots \theta_{t_n}$ is *a morphism of f-type* if there exists a sequence $\bar{i} = (i_1, \ldots, i_k)$ such that

$$g = {}^{i_k} f \circ \cdots \circ {}^{i_2} f \circ {}^{i_1} f.$$

Definition 3.12. Let $g, g_0, g_1, \ldots, g_m$ be morphisms in Soergel's category $\mathbf{B}$. We say that the tuple $(g_m, \ldots, g_1, g_0)$ is an *expression* of g if $g = g_m \circ \cdots \circ g_1 \circ g_0$.

Let $\bar{s} = (s_1, \ldots, s_n) \in \mathcal{S}^n$. We denote by $\theta_{\bar{s}}$ the bimodule $\theta_{s_1} \cdots \theta_{s_n}$. Let $(i_k, \ldots, i_2, i_1) \in \mathbb{N}^k$. We denote by $\mathrm{Null}(i_k, \ldots, i_1)$ the set $\{j \mid 1 \leqslant j \leqslant k \text{ and } i_j = 0\}$.

Proposition 3.13. *Let W be extra-large, i.e. $m(s,r) > 3$ for all $s, r \in \mathcal{S}$. Fix an integer $p \geqslant 2$. Let us consider $\bar{t} = (t_1, \ldots, t_{p-1})$ and $\bar{a} = (a_1, \ldots, a_{p-1})$, two reduced expressions of $x \in W$. Let $g \in \mathrm{Hom}(\theta_s\theta_{\bar{t}}, \theta_s\theta_{\bar{a}})$ be an f-type morphism. We have the following inclusion $g(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq 1 \otimes_{R^s} \theta_{\bar{a}}$.*

Proof. Let $g = {}^{i_k} f \circ \cdots \circ {}^{i_2} f \circ {}^{i_1} f$. If we consider the domain and the co-domain of g, we can conclude that the set $\mathrm{Null}(i_k, \ldots, i_1)$ has even cardinality.

Let $\mathrm{Null}(i_k, \ldots, i_1) = \{y_1, y_2, \ldots, y_{2\alpha}\}$ with $y_1 < y_2 < \cdots < y_{2\alpha}$ and $\alpha \geqslant 0$. We will prove the proposition by double induction in p and in α. We will use the notation $T(p_0, \alpha_0)$ if the proposition is true for $p = p_0$ and $\alpha = \alpha_0$. So we have to prove

(1) $T(i,0)$ for all $i \geqslant 2$.
(2) $(T(i,\mathbb{N})$ for all $2 \leqslant i < p) \Rightarrow$ (for all $\alpha \geqslant 0$, $T(p,\alpha) \Rightarrow T(p,\alpha+1))$,

where $\mathbb{N}$ is the set of non-negative integers.

The first assertion is trivial. We will prove the second assertion. We suppose $T(i,\mathbb{N})$ for all $2 \leqslant i < p$ and $T(p,\alpha)$, and we will prove $T(p,\alpha+1)$. We define $w = \begin{cases} t_1 & \text{if } m(s,t_1) \text{ is odd,} \\ s & \text{if } m(s,t_1) \text{ is even.} \end{cases}$

We define the three morphisms H, F, G by the formula

$$H = {}^{i_{y_2}}f \circ \cdots \circ {}^{i_2}f \circ {}^{i_1}f = {}^0f \circ F \circ {}^0f \circ G.$$

We define $\bar{t'}$, $\bar{t''}$, $\bar{a'}$, $\bar{a''}$ in the following way:

- $G \in \operatorname{Hom}(\theta_s\theta_{\bar{t}}, \theta_s\theta_{\bar{t'}})$.
- $F \in \operatorname{Hom}(\underbrace{\theta_{t_1}\theta_s\theta_{t_1}\cdots\theta_w}_{m(s,r)\text{ terms}}\theta_{\bar{t''}}, \underbrace{\theta_{t_1}\theta_s\theta_{t_1}\cdots\theta_w}_{m(s,r)\text{ terms}}\theta_{\bar{a'}})$.
- $H \in \operatorname{Hom}(\theta_s\theta_{\bar{t}}, \theta_s\theta_{\bar{a''}})$.

It is clear that $G(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq 1 \otimes_{R^s} \theta_{\bar{t'}}$. By Corollary 3.10,

$${}^0f \circ G(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq R^s \otimes_{R^{t_1}} 1 \otimes_{R^s} 1 \otimes_{R^{t_1}} \cdots 1 \otimes_{R^w} \theta_{\bar{t''}}.$$

By induction hypothesis, $T(i,\mathbb{N})$ for all $2 \leqslant i < p$, so in particular $T(p - m(s,t_1), \mathbb{N})$, and this implies

$$F \circ {}^0f \circ G(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq R^s \otimes_{R^{t_1}} 1 \otimes_{R^s} 1 \otimes_{R^{t_1}} \cdots 1 \otimes_{R^w} \theta_{\bar{a'}},$$

and finally $H(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq R^s \otimes_{R^s} \theta_{\bar{a''}} \subseteq 1 \otimes_{R^s} \theta_{\bar{a''}}$. Again by induction we know $T(p,\alpha)$, so if $H' = {}^{i_k}f \circ \cdots \circ {}^{i_{y_2+2}}f \circ {}^{i_{y_2+1}}f$, we have $H'(1 \otimes_{R^s} \theta_{\bar{a''}}) \subseteq 1 \otimes_{R^s} \theta_{\bar{a}}$, so

$$g(1 \otimes_{R^s} \theta_{\bar{t}}) = H' \circ H(1 \otimes_{R^s} \theta_{\bar{t}}) \subseteq 1 \otimes_{R^s} \theta_{\bar{a}}. \qquad \square$$

4. The bimodule E_w

From now on we will assume that our Coxeter group W is extra-large, *i.e.* $m(s,r) > 3$ for all $s, r \in \mathcal{S}$. In this section we fix an element $x \in W$ and a reduced expression $\bar{s} = (s_1, \ldots, s_n) \in \mathcal{S}^n$ of x.

4.1. Towards E_w

We start this section with some definitions. We say that an *integer interval* is a set of positive integers of the form $I = \{a, a+1, a+2, \ldots, b\}$. It might be the empty set. We will use the standard notation $I = [\![a,b]\!]$ (in this notation we assume $a \leqslant b$). If $a > b$ we define $[\![a,b]\!] = \emptyset$.

Let $\bar{r} = (r_1, \ldots, r_n) \in \mathcal{S}^n$ be a reduced expression of an element y of W. We denote by $\mathcal{P}(\{1,2,\ldots,n\})$ the power set of $\{1,2,\ldots,n\}$. We denote by $T(\bar{r})$ the subset of $\mathcal{P}(\{1,2,\ldots,n\})$ defined by $T(\bar{r}) = \{I \text{ integer interval} \mid \text{if } i, i+2 \in I, \text{ then } r_i = r_{i+2}\}$.

Consider the following union of subsets of $\mathcal{P}(\{1,2,\dots,n\})$,

$$T'(y)=\bigcup_{\bar{r}\in\mathcal{R}(y)} T(\bar{r})\subseteq\mathcal{P}\big(\{1,2,\dots,n\}\big),$$

where the symbol $\mathcal{R}(y)$ stands for the set of all reduced expressions of y. The elements of $\mathcal{P}(\{1,2,\dots,n\})$ are related by a partial order given by the inclusion. We define the set of integer intervals $A(y)$ as the subset of $T'(y)$ consisting of all maximal intervals with respect to the partial order given by inclusion.

Definition 4.2. If $A(y)=\bigcup_{j\in J}[\![a_j,b_j]\!]$ (a union of elements of $\mathcal{P}(\{1,2,\dots,n\})$), then we define:

- $\mathrm{Gcores}(y)=\bigcup_{j\in J}[\![a_j+1,b_j-1]\!]$, the set of *generalized cores*.
- $\mathrm{ELGcores}(y)=\{[\![a,b]\!]\in\mathrm{Gcores}(y)\mid b-a\geqslant 1\}$ the set of *extra-large generalized cores*.
- $\mathrm{cores}(y)=\{[\![a,b]\!]\in\mathrm{Gcores}(y)\mid m(r_a,r_{a+1})=b-a+3\}$, the set of *cores*, where $(r_1,\dots,r_n)$ is some reduced expression of y. It is easy to see that this set does not depend on the choice of the expression $(r_1,\dots,r_n)$. Moreover, $\mathrm{cores}(y)$ is easily seen to be the locations such that there exists a reduced expression where a braid relation can be applied.

4.3. If $y\in W$, we define a total order in the set $\mathrm{cores}(y)$ in the following way. If $C=[\![a,b]\!], C'=[\![a',b']\!]\in\mathrm{cores}(y)$, we say that $C<C'$ if and only if $b<a'$. We remark that it is a total order because W is extra-large. We define the distance from C to C' by

$$\mathrm{dist}\big(C,C'\big)=\min\big\{\big|a'-b\big|,\big|a-b'\big|\big\}-1.$$

We recall that in this section we fixed $\bar{s}=(s_1,\dots,s_n)\in\mathcal{S}^n$ a reduced expression of $x\in W$.

Definition 4.4. If $C=[\![a,b]\!]\in\mathrm{cores}(x)$ we define $\mathrm{first}(C)=a$ and $\mathrm{last}(C)=b$. The core C is called

- *right core* if $s_{b-1}\neq s_{b+1}$ and $s_{a-1}=s_{a+1}$,
- *left core* if $s_{b-1}=s_{b+1}$ and $s_{a-1}\neq s_{a+1}$,
- *empty core* if $s_{b-1}\neq s_{b+1}$ and $s_{a-1}\neq s_{a+1}$,
- *filled core* if $s_{b-1}=s_{b+1}, s_{a-1}=s_{a+1}$.

Remark 4.5. If $C<C'$, $\mathrm{dist}(C,C')=1$ and C is a right or empty core, then C' must be a right or filled core.

The following lemma is straightforward.

Lemma 4.6. *Let* $\mathrm{cores}(x)=\{C_1,C_2,\dots,C_k\}$ *with* $C_1<C_2<\dots<C_k$, *and let* $g\in\mathrm{End}(\theta_{\bar{s}})$ *be of* f*-type. Suppose that* $\mathrm{dist}(C_i,C_{i+1})\geqslant 2$ *for some* $1\leqslant i\leqslant k$. *If* $\mathrm{last}(C_i)=d-1$, $\bar{s}'=(s_1,\dots,s_d)$, $\bar{s}''=(s_{d+1},\dots,s_n)$, *then there exist* $g'\in\mathrm{End}(\theta_{\bar{s}'})$, $g''\in\mathrm{End}(\theta_{\bar{s}''})$, *both of* f*-type, such that* $g=g'\otimes g''$.

The following proposition is very important for the sequel.

Proposition 4.7. *Let $C', C'' \in \mathrm{cores}(x)$, $C' < C''$, $\mathrm{dist}(C', C'') = 1$ and suppose that $s_{d-2} = s_d = s_{d+2}$, for $d = \mathrm{last}(C') + 1$. Let us put $\bar{s}' = s_1 \cdots s_d$, $\bar{s}'' = s_d \cdots s_n$. If $g \in \mathrm{End}(\theta_{\bar{s}})$ is of f-type, there exist $g' \in \mathrm{End}(\theta_{\bar{s}'})$ and $g'' \in \mathrm{End}(\theta_{\bar{s}''})$, both of f-type, such that*

$$g = \left(g' \otimes \mathrm{id}^{n-d+1}\right) \circ \left(\mathrm{id}^d \otimes g''\right) = \left(\mathrm{id}^d \otimes g''\right) \circ \left(g' \otimes \mathrm{id}^{n-d+1}\right). \tag{4.1}$$

Remark 4.8. We remark that it is equivalent to say $s_{d-2} = s_d = s_{d+2}$ or to say that C' is a left or a filled core and C'' is a right or a filled core.

Proof. Let $b + 1 = \mathrm{first}(C')$ and $g = {}^{i_k}f \circ \cdots \circ {}^{i_2}f \circ {}^{i_1}f$. Consider the set $X(C', C'') = \{1 \leqslant p \leqslant k \mid i_p = b - 1 \text{ or } i_p = d - 1\}$ and let $X(C', C'') = \{x_1, x_2, \ldots, x_{2t}\}$ with $x_1 < x_2 < \cdots < x_{2t}$. As $s_{d-2} = s_d = s_{d+2}$, we can easily deduce by induction on l that $i_{x_{2l}} = i_{x_{2l-1}}$ for $1 \leqslant l \leqslant t$, and that is the reason why $X(C', C'')$ has an even number of elements. We define $x_0 = 0$ and $x_{2t+2} = k + 1$. For $0 \leqslant c \leqslant t$ we define the sets

$$Z_c^{<} = \{p \mid x_{2c} < p \leqslant x_{2c+2}\},$$
$$Z_c^{<} = Z_c \cap \{p \mid i_p < d - 1\}$$

and

$$Z_c^{\geqslant} = Z_c \cap \{p \mid i_p \geqslant d - 1\}.$$

For $1 \leqslant c \leqslant t$ we define $L_c = {}^{i_\alpha}f \circ {}^{i_\beta}f \circ \cdots \circ {}^{i_\gamma}f$ where the elements of $Z_c^{<}$ are $\alpha < \beta < \cdots < \gamma$, and $R_c = {}^{i_\delta}f \circ {}^{i_\epsilon}f \circ \cdots \circ {}^{i_\omega}f$ where the elements of $Z_c^{\geqslant}$ are $\delta < \epsilon < \cdots < \omega$.

It is clear that if $z \in Z_c^{<}$ and $w \in Z_c^{\geqslant}$ then ${}^{i_z}f$ commutes with ${}^{i_w}f$, so

$$L_c \circ R_c = R_c \circ L_c = {}^{i_\rho}f \circ \cdots \circ {}^{i_\sigma}f \circ {}^{i_\tau}f,$$

where the elements of Z_c are $\rho < \sigma < \cdots < \tau$.

We define $M = \theta_{s_1}\theta_{s_2} \cdots \theta_{s_{d-1}}$ and $N = \theta_{s_{d+1}} \cdots \theta_{s_n}$. So we have $\theta_{\bar{s}} = M \otimes_{R^{s_d}} N$. By Proposition 3.13 we have that for all $0 \leqslant c \leqslant t$ if $m \otimes n \in M \otimes_{R^{s_d}} N$, then $L_c(m \otimes n) \subseteq M \otimes n$ and $R_c(m \otimes n) \subseteq m \otimes N$, so for all $0 \leqslant a, c \leqslant t$ we have

$$R_a \circ L_c = L_c \circ R_a. \tag{4.2}$$

If we define $g' = L_t \circ L_{t-1} \circ \cdots \circ L_0$ and $g'' = R_t \circ R_{t-1} \circ \cdots \circ R_0$ then Eq. (4.2) allows us to finish the proof. $\square$

Before we state the next theorem we have to make a definition.

Definition 4.9. Let $C \in \mathrm{cores}(x)$. We say that a tuple $\bar{\alpha} = (i_k, \ldots, i_2, i_1) \in \mathbb{N}^k$, $k \in \mathbb{N}$ is $\bar{s}$-*compatible* if the morphism $g = {}^{i_k}f \circ \cdots \circ {}^{i_2}f \circ {}^{i_1}f \in \mathrm{End}(\theta_{\bar{s}})$ is well defined, and in this case we say that g is the morphism associated with $\bar{\alpha}$. If $\bar{\alpha}$ is $\bar{s}$-compatible we define $N_{\bar{\alpha}}(C) = \mathrm{card}(\{p \mid i_p = \mathrm{first}(C) - 2\})$, where card stands for cardinality.

Theorem 4.10. *Let $\bar{\alpha}$ and $\bar{\beta}$ be two $\bar{s}$-compatible tuples, and g, h the morphisms associated with $\bar{\alpha}$ and $\bar{\beta}$ respectively. We have that $g = h$ if and only if for all $C \in \mathrm{cores}(x)$ we have $N_{\bar{\alpha}}(C) = 0 \Leftrightarrow N_{\bar{\beta}}(C) = 0$.*

Proof. The "only if" part is evident. We will prove the "if" part. Let $\mathrm{cores}(x) = \{C_1, \dots, C_k\}$ with $C_1 < C_2 < \cdots < C_k$. We will prove the theorem by induction over k.

It is easy to see using the definition of f_{sr} that $f_{sr} \circ f_{rs} \circ f_{sr} = f_{sr}$ and this allows us to prove the theorem for $k = 1$.

We suppose the theorem is true for $k-1$ and we will prove it for k. If there exists $1 \leqslant i \leqslant k-1$ such that $\mathrm{dist}(C_i, C_{i+1}) \geqslant 2$, then Lemma 4.6 and the induction hypothesis allows us to conclude. So we suppose $\mathrm{dist}(C_i, C_{i+1}) = 1$ for all $1 \leqslant i \leqslant k-1$.

If there exists $1 \leqslant i \leqslant k$ such that $N_{\bar{\alpha}}(C_i) = 0$ we can conclude by induction, so we suppose $N_{\bar{\alpha}}(C_i) \neq 0$ for all $1 \leqslant i \leqslant k$.

Let $(j_i)_{1 \leqslant i \leqslant \gamma}$ be an ascending sequence of numbers such that the set of filled cores is exactly $\{C_{j_i}\}_{1 \leqslant i \leqslant \gamma}$. By Proposition 4.7, Remark 4.8 and the fact that $\mathrm{dist}(C_i, C_{i+1}) = 1$ we see that if C_i is a filled core, then C_{i+1} cannot be a filled core, so $j_{i+1} - j_i \geqslant 2$. We can conclude that the set $\mathcal{R}_i = \{j_i + 1, j_i + 2, \dots, j_{i+1} - 1\}$ is non-empty for all $1 \leqslant i < \gamma$.

Let us suppose that there exists an integer p such that for all $i \in \mathcal{R}_p$ the core C_i is not an empty core. So if $i \in \mathcal{R}_p$, then C_i is either left or right core. By Remark 4.5, we know that there exists an integer v, with $j_p + 1 \leqslant v \leqslant j_{p+1}$, such that if $j_p + 1 < i \leqslant v$ then C_i is a left core and if $v < i < j_{p+1}$ then C_i is a right core. We are in the case treated in Proposition 4.7 with $C' = C_v$ and $C'' = C_{v+1}$ (see Remark 4.8), so we can conclude by induction.

So from now on we suppose that for all integers $1 \leqslant p < \gamma$ there exists some $i_p \in \mathcal{R}_p$ such that the core C_{i_p} is an empty core. By Remark 4.5, if $j_p < i < i_p$ then C_i is a left core and if $i_p < i < j_{p+1}$, then C_i is a right core. Then i_p is well defined for $1 \leqslant p < \gamma$.

Let $\bar{\alpha} = (\alpha_w, \dots, \alpha_1)$, so $g = {}^{\alpha_w}f \circ \cdots \circ {}^{\alpha_2}f \circ {}^{\alpha_1}f$. If we consider the set of all $\bar{s}$-compatible tuples satisfying that g is their associated morphism (see Definition 4.9), with the partial order $\stackrel{\bullet}{\leqslant}$ given by the length of the tuple, we can assume without loss of generality that $\bar{\alpha}$ and $\bar{\beta}$ are minimal for $\stackrel{\bullet}{\leqslant}$.

Lemma 4.11. *For all $1 \leqslant p < \gamma$ we have $N_{\bar{\alpha}}(C_{j_p}) = 2$ (recall that C_{j_p} is a filled core).*

Proof. Let us suppose that this is false, so there exists some $N_{\bar{\alpha}}(C_{j_p}) \geqslant 4$. By definition this means that the set $Z = \{1 \leqslant i \leqslant w \mid \alpha_i = \mathrm{first}(C_{j_p}) - 2\}$ has more than three elements. We denote $a = \mathrm{first}(C_{j_p}) - 2$. Let $z_1 < z_2 < z_3$ be the first three elements of Z. As by definition C_{j_p} is a filled core, we have that $C_{j_p - 1}$ is either a right or an empty core and $C_{j_p + 1}$ is either a left or an empty core, so we can conclude that ${}^{\alpha_{z_3}}f$ commutes with ${}^{\alpha_{z_3 - 1}}f \circ \cdots \circ {}^{\alpha_{z_2+2}}f \circ {}^{\alpha_{z_2+1}}f$, so by commuting this two morphisms we get

$$g = {}^{\alpha_w}f \circ \cdots \circ \widehat{{}^{\alpha_{z_3}}f} \circ \cdots \circ {}^{\alpha_{z_2+2}}f \circ {}^{\alpha_{z_2+1}}f \circ \left({}^{a}f \circ {}^{a}f\right) \circ {}^{\alpha_{z_2-1}}f \circ \cdots \circ {}^{\alpha_2}f \circ {}^{\alpha_1}f$$

where $\widehat{{}^{\alpha_{z_3}}f}$ means that we skip this term. Because of Proposition 3.13, $({}^{a}f \circ {}^{a}f)$ commutes with ${}^{\alpha_{z_2-1}}f \circ \cdots \circ {}^{\alpha_{z_1+2}}f \circ {}^{\alpha_{z_1+1}}f$, so again by commuting this two morphisms we get

$$g = {}^{\alpha_w}f \circ \cdots \circ \widehat{{}^{\alpha_{z_3}}f} \circ \cdots \circ \widehat{{}^{\alpha_{z_2}}f} \circ \cdots \circ {}^{\alpha_{z_1+1}}f \circ \left({}^{a}f \circ {}^{a}f \circ {}^{a}f\right) \circ {}^{\alpha_{z_1-1}}f \circ \cdots \circ {}^{\alpha_2}f \circ {}^{\alpha_1}f$$

but the fact that ${}^{a}f \circ {}^{a}f \circ {}^{a}f = {}^{a}f$ contradicts the minimality of $\bar{\alpha}$ in the $\lessdot$ order. So this proves Lemma 4.11. □

Let us recall some of the notation we have introduced throughout this proof. Recall that $\bar{s} = (s_1, \ldots, s_n)$, $g = {}^{\alpha_w}f \circ \cdots \circ {}^{\alpha_2}f \circ {}^{\alpha_1}f$, $\mathrm{cores}(x) = \{C_1, \ldots, C_k\}$, the set of filled cores is $\{C_{j_i}\}_{1 \leqslant i \leqslant \gamma}$ and the set of empty cores is $\{C_{i_p}\}_{1 \leqslant p < \gamma}$.

Notation 4.12. For $1 \leqslant i \leqslant k$ we put $f(C_i) = {}^{y}f$ with $y = \mathrm{first}(C_i) - 2$. We define $i_\gamma = n$. For $1 \leqslant p \leqslant \gamma + 1$, we consider the set

$$V_p^{\mathrm{left}} = \bigl\{1 \leqslant i \leqslant w \bigm| {}^{\alpha_i}f = f(C_q) \text{ for } j_p < q \leqslant i_p\bigr\},$$

and put $V_p^{\mathrm{left}} = \{v_1, v_2, \ldots, v_{u(p)}\}$ with $v_1 < v_2 < \cdots < v_{u(p)}$. We define a sequence $\bar{\alpha}^{\mathrm{left}}(p) = (n_1, n_2, \ldots, n_{u(p)})$ by putting $n_i = q - j_p$ if ${}^{\alpha_{v_i}}f = f(C_q)$.

Lemma 4.13. *For $1 \leqslant p \leqslant \gamma$ we have*

$$\bar{\alpha}^{\mathrm{left}}(p) = \bigl(1, 2, \ldots, u(p)/2, u(p)/2, \ldots, 2, 1\bigr), \tag{4.3}$$

where $u(p)/2 = i_p - j_p$.

Proof. Let us prove by induction in l that $(n_1, n_2, \ldots, n_l) = (1, 2, \ldots, l)$ for $1 \leqslant l \leqslant u(p)/2$ (the proof of $(n_{u(p)/2+1}, n_{u(p)/2+2}, \ldots, n_{u(p)/2+l}) = (u(p)/2, \ldots, u(p)/2 - l + 1)$ is similar). As C_i is a left core for $j_p < i < i_p$, it is clear that $n_1 = 1$. Let us suppose $(n_1, n_2, \ldots, n_{l-1}) = (1, 2, \ldots, l - 1)$ for $1 \leqslant l \leqslant u(p)/2$, we will prove that $n_l = l$. If this is not the case, we must have $n_l = l - 1$. We now have two possibilities for n_{l+1}: it can be $l - 1$ or $l - 2$. But if $n_{l+1} = l - 1$, with commutations relations we will obtain a subexpression of the form ${}^{a}f \circ {}^{a}f \circ {}^{a}f$, for some integer a, and this contradicts the minimality of $\bar{\alpha}$ in the $\lessdot$ order. So we conclude that $n_{l+1} = l - 2$.

As C_i is a left core for $j_p < i < i_p$ we have that $|n_i - n_{i+1}| \leqslant 1$ for all i, and the hypothesis we made in this proof that $N_{\bar{\alpha}}(C_i) \neq 0$ for all $1 \leqslant i \leqslant k$ allows us to conclude that $\{n_i\}_{1 \leqslant i \leqslant u(p)} = \{1, 2, \ldots, u(p)/2\}$. So consider m the minimum of the set $\{i > l \mid n_i = l - 1\}$. Because of Proposition 3.13 we deduce that with commutation relations we can obtain an expression of g with a subexpression of the form ${}^{\alpha_{v_{l-1}}}f \circ {}^{\alpha_{v_{l-2}}}f \circ {}^{\alpha_{v_m}}f$, which again contradicts the minimality of $\bar{\alpha}$ in the $\lessdot$ order thus proving the lemma. □

We now repeat Notation 4.12 but changing left by right.

Notation 4.14. We define $i_0 = 1$. We now consider, for $0 \leqslant p \leqslant \gamma$

$$V_p^{\mathrm{right}} = \bigl\{1 \leqslant i \leqslant w \bigm| {}^{\alpha_i}f = f(C_q) \text{ for } i_p \leqslant q < j_{p+1}\bigr\},$$

and put $V_p^{\mathrm{right}} = \{d_1, d_2, \ldots, d_{e(p)}\}$ with $d_1 < d_2 < \cdots < d_{e(p)}$. We define a sequence $\bar{\alpha}^{\mathrm{right}}(p) = (m_1, m_2, \ldots, m_{e(p)})$ by putting $m_q = j_{p+1} - q$ if ${}^{\alpha_{d_i}}f = f(C_q)$.

By similar arguments as in Lemma 4.13 we conclude that

$$\bar{\alpha}^{\text{right}}(p) = \big(1, 2, \ldots, e(p)/2, e(p)/2, \ldots, 2, 1\big), \tag{4.4}$$

where $e(p)/2 = j_{p+1} - i_p$.

The two equations (4.3) and (4.4) allow us to conclude that for all $1 \leqslant p \leqslant \gamma + 1$

$$\bar{\alpha}^{\text{left}}(p) = \bar{\beta}^{\text{left}}(p) = \big(1, 2, \ldots, u(p)/2, u(p)/2, \ldots, 2, 1\big) \tag{4.5}$$

and for all $0 \leqslant p \leqslant \gamma$

$$\bar{\alpha}^{\text{right}}(p) = \bar{\beta}^{\text{right}}(p) = \big(1, 2, \ldots, e(p)/2, e(p)/2, \ldots, 2, 1\big). \tag{4.6}$$

To finish the proof of Theorem 4.10, we will only need Eqs. (4.5) and (4.6).

Before we can prove Theorem 4.10 we need some definitions.

Definition 4.15. We will say that a morphism $\delta \in \operatorname{Hom}(\theta_{t_1} \cdots \theta_{t_p}, \theta_{r_1} \cdots \theta_{r_p})$ is a *p-morphism* if there exists a sequence of integers $(a_0, \ldots, a_p)$ such that $\delta = \mathrm{id}^{a_0} \otimes f \otimes \mathrm{id}^{a_1} \otimes f \otimes \cdots \otimes \mathrm{id}^{a_{p-1}} \otimes f \otimes \mathrm{id}^{a_p}$.

Let $g = (g_m, g_{m-1}, \ldots, g_1)$ and $g' = (g'_d, g'_{d-1}, \ldots, g'_1)$ be two expressions of the same morphism (recall Definition 3.12), such that g_i is a p_i-morphism and g'_i is a p'_i-morphism. We say that $g < g'$ if and only if $(p_1, \ldots, p_m) < (p'_1, \ldots, p'_d)$ in the lexicographical order. We will call this order the "morphism" order.

Definition 4.16. Consider two morphisms $g : M_1 \to N_1$ and $f : M_2 \to N_2$. The relation $(f \otimes \mathrm{id}_{N_1}) \circ (\mathrm{id}_{M_2} \otimes g) = (\mathrm{id}_{N_2} \otimes g) \circ (f \otimes \mathrm{id}_{m_1})$, satisfied in all tensor categories, will be called *commutation relation*.

We choose $\bar{g} = (g_m, g_{m-1}, \ldots, g_0)$ an expression of g and $\bar{h} = (h_d, h_{d-1}, \ldots, d_0)$ an expression of h, both maximal in the morphism order. We will prove that $d = m$ and that $g_i = h_i$ for all $0 \leqslant i \leqslant m$. In fact we will prove more; we will explicitly determine all the g_i.

We will say that a p-morphism ω acts on the cores $T = \{C_{n_1}, C_{n_2}, \ldots, C_{n_p}\}$ if ω does not act as the identity exactly in that set of cores, and evidently ω is determined by T.

We put $y_p = \max\{i_p - j_p, j_{p+1} - i_p\}$. For all $i \geqslant 0$ for which this definition is not empty we define the following sets:

$$T_i^{1,p} = C_{j_p + i < i_p} \cup C_{j_p - i > i_{p-1}},$$

$$T_i^{2,p} = \delta_{i, y_p} \cdot C_{i_p},$$

$$T_i^{3,p} = \delta_{i, y_p + 1} \cdot C_{i_p},$$

$$T_i^{4,p} = C_{i_p + y_p - i > j_p} \cup C_{i_p + i - y_p < j_{p+1}}$$

and

$$T_i = \bigcup_{1 \leqslant p \leqslant \gamma} T_i^{1,p} \cup T_i^{2,p} \cup T_i^{3,p} \cup T_i^{4,p}.$$

It is easy to see that if ω_i acts on T_i, then the expression $\overline{\omega} = (\omega_m, \omega_{m-1}, \dots, \omega_0)$ is maximal in the morphism order and satisfies Eqs. (4.5) and (4.6). So to prove Theorem 4.10 we only need to prove that with commutation relations we can pass from $\bar{g}$ to $\overline{\omega}$. We will prove the following property by induction on i.

Property $(*)$: With commutation relations we can pass from $\bar{g}$ to an expression of the form $({}^{b_n}f, \dots, {}^{b_1}f, \omega_i, \dots, \omega_0)$, with $b_1, \dots, b_n \in \mathbb{Z}$.

We have that ω_0 is by definition the morphism that acts on the set $\{C_{j_p}\}_{1 \leqslant p \leqslant \gamma}$, so property $(*)$ is clear for $i = 0$.

Let us suppose that property $(*)$ is true for i, we will prove it for $i+1$. So we have an expression $({}^{b_n}f, \dots, {}^{b_1}f, \omega_i, \dots, \omega_0)$. Let us consider $1 \leqslant p \leqslant \gamma$. We have to consider four cases:

(1) $i < y_p$.
(2) $i = y_p$.
(3) $i = y_p + 1$.
(4) $i > y_p$.

Let $1 \leqslant a \leqslant 4$. By Eqs. (4.5) and (4.6) we see that in case a we can make commutation relations in the subexpression $({}^{b_n}f, \dots, {}^{b_1}f)$, and arrive at an expression of the form $({}^{b'_u}f, \dots, {}^{b'_1}f, \omega_{i+1}^p)$, where ω_{i+1}^p is a p-morphism that acts exactly on $T_{i+1}^{a,p}$. As we can do this for all p, we can go with commutation relations from $({}^{b_n}f, \dots, {}^{b_1}f)$ to an expression of the form

$$\left({}^{b''_v}f, \dots, {}^{b''_1}f, \omega_{i+1}^{\gamma}, \dots, \omega_{i+1}^2, \omega_{i+1}^1\right).$$

The fact that $(\omega_{i+1}^{\gamma}, \dots, \omega_{i+1}^2, \omega_{i+1}^1)$ is an expression of the morphism ω_{i+1} allows us to finish the proof of property $(*)$ and of Theorem 4.10. □

5. The idempotents

Let $\bar{s} = (s_1, \dots, s_n) \in \mathcal{S}^n$ be a reduced expression of $x \in W$. We can now define a special morphism $f_{\bar{s}} = {}^{i_k}f \circ \cdots \circ {}^{i_2}f \circ {}^{i_1}f \in \mathrm{End}(\theta_{\bar{s}})$. It is the morphism characterized by the fact that, if $\bar{i} = (i_k, \dots, i_1)$, then $N_{\bar{i}}(C) \neq 0$ for all $C \in \mathrm{cores}(x)$. Theorem 4.10 shows that if this morphism exists, it is unique. Now we will show that at least one such morphism exists.

We know that we can pass from any reduced expression $\bar{r}$ of x to any other $\bar{t}$, by a sequence of braid relations. This induces a morphism of f-type in $\mathrm{Hom}(\theta_{\bar{r}}, \theta_{\bar{t}})$. If we make a sequence of braid relations starting in $\bar{s}$, ending in $\bar{s}$ and passing through all reduced expressions of x in any way we want, the corresponding morphism satisfies the requirements for $f_{\bar{s}}$. So we conclude that $f_{\bar{s}}$ is a well-defined morphism.

Theorem 4.10 tells us that $f_{\bar{s}}^2 = f_{\bar{s}}$. This means that $f_{\bar{s}}$ is an idempotent and so we conclude that the bimodule $f_{\bar{s}}(\theta_{\bar{s}})$ (that we denote by $E_{\bar{s}}$) is an element of Soergel's category.

Theorem 5.1. *If $\bar{r}$ and $\bar{t}$ are two reduced expressions of the same element $y \in W$ then $E_{\bar{r}}$ is isomorphic to $E_{\bar{t}}$.*

Proof. We need to find an isomorphism between $E_{\bar{r}}$ and $E_{\bar{t}}$ for any two reduced expressions $\bar{r}$ and $\bar{t}$ of the element $y \in W$. As we can pass from any reduced expression of y to any other one by a sequence of braid relations, it is enough to find an isomorphism between $E_{\bar{r}}$ and $E_{\bar{t}}$ when

$\bar{r}$ and $\bar{t}$ differ only by one braid relation. Let us call the associated morphism $F_{\bar{r},\bar{t}} : \theta_{\bar{r}} \to \theta_{\bar{t}}$. We define $i_{\bar{r}} : E_{\bar{r}} \to \theta_{\bar{r}}$, the natural inclusion. We define the projection $f'_{\bar{t}} : \theta_{\bar{t}} \to E_{\bar{t}}$ to be the same as $f_{\bar{t}} : \theta_{\bar{t}} \to \theta_{\bar{t}}$, up to the fact that we restrict the target. We define $a_{\bar{r},\bar{t}}$ as follows.

$$\begin{array}{ccc} \theta_{\bar{r}} & \xrightarrow{F_{\bar{r},\bar{t}}} & \theta_{\bar{t}} \\ {\scriptstyle i_{\bar{r}}}\uparrow & & \downarrow{\scriptstyle f'_{\bar{t}}} \\ E_{\bar{r}} & \xrightarrow[a_{\bar{r},\bar{t}}]{} & E_{\bar{t}} \end{array}$$

To finish the proof of Theorem 5.1 we only need to prove

$$a_{\bar{t},\bar{r}} \circ a_{\bar{r},\bar{t}} = \mathrm{id}_{E_{\bar{r}}},$$

so we only need to prove

$$f_{\bar{r}} \circ F_{\bar{t},\bar{r}} \circ f_{\bar{t}} \circ F_{\bar{r},\bar{t}} \circ f_{\bar{r}} = f_{\bar{r}}, \tag{5.1}$$

but this is a direct consequence of Theorem 4.10 and the definition of $f_{\bar{r}}$. □

Notation 5.2. If $\bar{r}$ is a reduced expression of $w \in W$ we define $E_w = E_{\bar{r}}$, and this is a well-defined bimodule up to isomorphism.

5.3. Let us consider the fraction field K of R. Let K_w denote the (K, K)-bimodule equal to K as left module and with the right action of K twisted by w. In formulas this is $k \cdot k' = kw(k')$, for $k, k' \in K$. We recall that $\eta : \langle \mathbf{B} \rangle \to \mathcal{H}$ is the inverse of ε.

Lemma 5.4. *Let M be a Soergel bimodule and let the polynomials $p_w \in \mathbb{Z}[v, v^{-1}]$ be defined by $\eta(\langle M \rangle) = \sum_{w \in W} p_w T_w$. We have the following formula:*

$$K \otimes_R M \cong \bigoplus_{w \in W} p_w(1) K_w.$$

Proof. As (V, V) is a good couple (bonne paire), by [8, formule (3.6)] we have that $K \otimes_R \theta_s \cong K \oplus K_s$. If $(s_1, \ldots, s_n)$ is a sequence of elements of $\mathcal{S}$ we have that

$$K \otimes_R \theta_{s_1} \cdots \theta_{s_n} \cong (K \oplus K_{s_1}) \otimes_R \cdots \otimes_R (K \oplus K_{s_n}).$$

By definition of ε, we have $\eta(\langle \theta_{s_1} \cdots \theta_{s_n} \rangle) = (1 + T_{s_1}) \cdots (1 + T_{s_n})$. To specialize this element in $q = 1$ is equivalent to calculate $(1 + s_1) \cdots (1 + s_n)$ in the group algebra $k[W]$ (identifying s_i with T_{s_i}), so the fact that $K_x \otimes_R K_y \cong K_{xy}$ for all $x, y \in W$, allows us to finish the proof of the lemma for $M = \theta_{s_1} \cdots \theta_{s_n}$.

It is trivial to extend this result to finite direct sums of bimodules of the form $\theta_{s_1} \cdots \theta_{s_n}$, and for the direct summands it is enough to use the characterization of Soergel bimodules given in [12, Lemma 5.13]. □

Notation 5.5. In [12] Soergel classifies the indecomposable bimodules in **B** and for each $x \in W$ he defines an indecomposable bimodule B_x satisfying some support properties. We denote by B'_x the shifted indecomposable bimodule $B_x(-l(x))$. For our purposes we only need to know that B'_x is indecomposable and that it is a direct summand of every product $\theta_{s_1} \cdots \theta_{s_n}$ such that $(s_1, \ldots, s_n)$ is a reduced expression of x.

Proposition 5.6. *The indecomposable Soergel bimodule B'_w is a direct summand of E_w and the set $\{\eta(\langle E_w \rangle)\}_{w \in W}$ is a basis of the Hecke algebra.*

Proof. Let us recall that K be the fraction field of R. Let $s, r \in \mathcal{S}$ and $z = \underbrace{srs\cdots}_{m(s,r)\text{ times}}$. In this proof we will use the following notation. If M is a Soergel bimodule, $\overline{M} = K \otimes_R M$ is the corresponding (K, K)-bimodule (see [8, Lemme 3.4]). We can find a projection $\pi : X_{sr} \to B'_z$ and an injection $in : B'_z \to X_{rs}$ such that $in \circ \pi = f_{sr} \in \operatorname{Hom}(X_{sr}, X_{rs})$. We note that, up to a scalar, we have $\pi = \Phi$ and $in = \Psi$, with Φ and Ψ as defined in Section 3.

By Lemma 5.4 we know that K_z is a direct summand of $\overline{X_{sr}}, \overline{X_{rs}}$, and appears in both bimodules with multiplicity one. To see that K_z is a direct summand of $\overline{B'_z}$ we also need the known fact that Soergel's conjecture is true for dihedral groups, so $\varepsilon(v^{-n} C'_z) = \langle B'_z \rangle$ and $C'_z = v^n(\sum_{y \leqslant z} T_y)$ (see [12, Remark 4.4]). So we have an injection $j : K_z \hookrightarrow \overline{X_{sr}}$. As $\mathrm{id}_K \otimes_R \pi : \overline{X_{sr}} \twoheadrightarrow \overline{B'_z}$ is a surjection and the space

$$\operatorname{Hom}_{(K,K)}(K_u, K_v) \simeq \begin{cases} \{0\} & \text{if } u \neq v, \\ K & \text{if } u = v, \end{cases}$$

we deduce an injection $(\mathrm{id}_K \otimes_R \pi) \circ j : K_z \hookrightarrow \overline{B'_z}$, and by composing with the injection $\mathrm{id}_K \otimes_R in$ (which is an injection because K is flat over R), we finally obtain an injection $\overline{f_{sr}} \circ j : K_z \hookrightarrow \overline{X_{rs}}$. This allows us to conclude that K_z is a direct summand of $\overline{f_{sr}}(\overline{X_{sr}}) = \overline{f_{sr}(X_{sr})}$. As $K_u \otimes_R K_v \cong K_{uv}$ for all $u, v \in W$, we conclude that K_u is a direct summand of $\overline{E_u}$, for all $u \in W$.

Let $\bar{s}$ be a reduced expression of w. We have that $\theta_{\bar{s}} = E_w \oplus Y = B'_w \oplus X$, for some Soergel bimodules X, Y, so

$$\overline{\theta_{\bar{s}}} = \overline{E_w} \oplus \overline{Y} = \overline{B'_w} \oplus \overline{X}.$$

By Krull–Schmidt (see [12, Remark 1.3]), B'_w is either direct summand of E_w or of Y (recall that B'_w is indecomposable). By [12, satz 6.16] we have that R_w is a submodule of B'_w, and as K is flat over R, then K_w is a direct summand of $\overline{B'_w}$. We have seen that K_w is a direct summand of $\overline{E_w}$ and of $\overline{\theta_{\bar{s}}}$, but it has multiplicity one in $\overline{\theta_{\bar{s}}}$ so B'_w is a direct summand of E'_w, thus proving the first part of the proposition.

By a theorem of Soergel (recalled in this paper in Corollary 6.2 of Section 6), if

$$\eta\big(\langle E_w \rangle\big) = \sum_v p_v T_v,$$

then $p_v \in \mathbb{N}[v, v^{-1}]$. As K_w appears with multiplicity one in $\overline{E_w}$ we conclude that $p_w = 1$, so $\eta(\langle E_w \rangle) = T_w + \sum_{v<w} p_v T_v$. This triangularity condition allows us to conclude the second part of the proposition. □

5.7. If we choose for every element $x \in W$ a reduced expression $(s_1^x, \dots, s_n^x)$ of x, then the set $\underline{Y} = \{C'_{s_1^x} \cdots C'_{s_n^x}\}_{x \in W}$ is a basis of the Hecke algebra.

Proposition 5.6 makes precise the assertion that the basis $\underline{A} = \{\eta(\langle E_w \rangle)\}_{w \in W}$ lies between the Kazhdan–Lusztig basis and the $\underline{Y}$ basis (Bott–Samelson–Demazure basis). We finish this section with an example that shows that $\underline{A}$ is different from both bases.

Example. Let $m(s, r) = 4$. By definition E_{srsr} is the image of $f_{rs} \circ f_{sr}$, and this is the indecomposable Soergel bimodule B'_{srsr}, so we conclude that $\eta(\langle E_{srsr} \rangle) = v^{-4} C'_{srsr}$. On the other hand, E_{srs} is by definition the bimodule $\theta_s \theta_r \theta_s$, and so

$$\eta\big(\langle E_{srs} \rangle\big) = v^{-3} C'_s C'_r C'_s \neq v^{-3} C'_{srs}.$$

In the next section we will get closer to the indecomposables considering some new morphisms.

6. The bimodule D_w

In this section we will construct a bimodule D_w in Soergel's category of bimodules, and we will study some of its properties. We start by recalling some of Soergel's results.

6.1. Notation

Given a $\mathbb{Z}$-graded vector space $V = \bigoplus_i V_i$, with $\dim(V) < \infty$, we define its graded dimension by the formula

$$\underline{\dim}\, V = \sum (\dim V_i) v^{-i} \in \mathbb{Z}\big[v, v^{-1}\big].$$

Let us recall that R_+ is the ideal of R generated by the homogeneous elements of non-zero degree. We define the graded rank of a finitely generated $\mathbb{Z}$-graded R-module

$$\underline{\mathrm{rk}}\, M = \underline{\dim}(M/MR_+) \in \mathbb{Z}\big[v, v^{-1}\big].$$

We have $\underline{\dim}(V(1)) = v(\underline{\dim} V)$ and $\underline{\mathrm{rk}}(M(1)) = v(\underline{\mathrm{rk}}\, M)$. We define $\overline{\underline{\mathrm{rk}}}\, M$ as the image of $\underline{\mathrm{rk}}\, M$ under $v \mapsto v^{-1}$.

For $x \in W$, we define the (R, R)-bimodule R_x as the set R with the usual left action but with the right action twisted by x (in formulas: $r \cdot r' = rx(r')$ for $r \in R_x$ and $r' \in R$).

Let us recall that there exists a unique involution d of the Hecke algebra with $d(v) = v^{-1}$ and $d(T_x) = (T_{x^{-1}})^{-1}$. An immediate corollary of Proposition 5.7, Proposition 5.9 and Corollary 5.16 of [12] is

Corollary 6.2 *(Soergel). We have the following equations in the Hecke algebra*

$$\begin{aligned}\eta\big(\langle B \rangle\big) &= \sum_{x \in W} \overline{\underline{\mathrm{rk}}}\, \mathrm{Hom}(B, R_x) T_x \\ &= d \circ \sum_{x \in W} \overline{\underline{\mathrm{rk}}}\, \mathrm{Hom}(R_x, B) T_x.\end{aligned}$$

6.3. The morphism $f^2_{sr}(n)$

In this subsection we prove that there exists a morphism generalizing $f_{rs} \circ f_{sr}$. For $s, r \in \mathcal{S}$ and $1 \leqslant n \leqslant m(s,r)$ we define $sr(n) = \underbrace{srs\cdots}_{n}$ and

$$\theta^l_{sr}(n) = \underbrace{\theta_s\theta_r\theta_s\cdots}_{n}, \qquad \theta^r_{sr}(n) = \underbrace{\cdots\theta_s\theta_r\theta_s}_{n}.$$

Proposition 6.4. *For $s, r \in \mathcal{S}$ and $1 \leqslant n \leqslant m(s,r)$ there exists a unique degree zero idempotent $f^2_{sr}(n)$ in $\operatorname{End}(\theta^l_{sr}(n))$ that factors through $B'_{sr(n)}$.*

Remark 6.5. The equation $f^2_{sr}(m(s,r)) = f_{rs} \circ f_{sr}$ explains the choice of our notation.

Proof. To prove this proposition we only need to prove the following two assertions:

- the space of degree zero morphisms in $\operatorname{Hom}(B'_{sr(n)}, \theta^l_{sr}(n))$ is one-dimensional,
- the space of degree zero morphisms in $\operatorname{Hom}(\theta^l_{sr}(n), B'_{sr(n)})$ is one-dimensional.

Using the adjunction [7, Lemme 3.3], this is equivalent to

(1) the space of morphisms of degree $-2n$ in $\operatorname{Hom}(\theta^r_{sr}(n)B'_{sr(n)}, R)$ is one-dimensional,
(2) the space of morphisms of degree $2n$ in $\operatorname{Hom}(R, \theta^r_{sr}(n)B'_{sr(n)})$ is one-dimensional.

We will prove the first claim. The second claim has a similar proof. Because of [12, Theorem 5.15], we know that the Hom spaces between Soergel bimodules are free as right R-modules, so we have $\operatorname{Hom}(\theta^r_{sr}(n)B'_{sr(n)}, R) \cong \bigoplus_i n_i R(2i)$ for some integers n_i. Let us see how to calculate these numbers. Let us define

$$h_0 = \underbrace{\cdots(1+T_s)(1+T_r)(1+T_s)}_{n \text{ terms}} C'_{sr(n)} v^{-n},$$

and let $\tau : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ be the map defined by

$$\tau\left(\sum_{x\in W} p_x T_x\right) = p_1 \quad \big(p_x \in \mathbb{Z}\big[v, v^{-1}\big]\big).$$

By Corollary 6.2 we have the equations

$$\begin{aligned}
\tau(h_0) &= \tau \circ \eta \circ \varepsilon(h_0) \\
&= \tau \circ \eta\big(\big\langle \theta^r_{sr}(n)B'_{sr(n)}\big\rangle\big) \\
&= \underline{\overline{\mathrm{rk}}}\operatorname{Hom}\big(\theta^r_{sr}(n)B'_{sr(n)}, R\big) \\
&= \underline{\overline{\mathrm{rk}}}\left(\bigoplus_i n_i R(2i)\right)
\end{aligned}$$

$$= \sum n_i v^{-2i} \, \underline{\overline{\mathrm{rk}}}\, R$$
$$= \sum n_i q^i.$$

So our problem reduces to an easy problem in the Hecke algebra, namely, to prove that $\tau(h_0) = q^n + \sum_{i<n} n_i q^i$. By [12, Remark 4.4] we have the formula $C'_{sr(n)} = v^n(\sum_{x \leqslant sr(n)} T_x)$ and by [4, Proposition 8.1.1] we know that

$$\tau(T_x T_{y^{-1}}) = \begin{cases} q^{l(x)} & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \tag{6.1}$$

The following lemma is easily proved by induction.

Lemma 6.6. *Let us define the polynomials $p_y \in \mathbb{Z}[q]$ by the formula*

$$\underbrace{\cdots(1+T_s)(1+T_r)(1+T_s)}_{n \text{ terms}} = \sum p_y T_y.$$

If $ys < y$ then $\deg(p_y) \leqslant (n - l(y))/2$, if $ys > y$ then $\deg(p_y) \leqslant (n - l(y) - 1)/2$ and $p_{sr(n)} = 1$.

This lemma allows us to finish the proof of Proposition 6.4. □

We recall Corollary 4.2 of [7].

Lemma 6.7. *Let $(s_1, \ldots, s_n) \in \mathcal{S}^n$. We define the integers m_i by $\tau((1 + T_{s_1}) \cdots (1 + T_{s_n})) = \sum_i m_i q^i$. Then, there exists an isomorphism of graded right R-modules*

$$\mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, R) \cong \bigoplus_i m_i R(2i).$$

Using Lemmas 6.6 and 6.7 we can conclude the following proposition.

Proposition 6.8. *Let $s, r \in \mathcal{S}$ and $1 \leqslant n \leqslant m(s,r)$. If $f \in \mathrm{Hom}(\theta^r_{sr}(n), R)$, then $\deg(f) \geqslant -2[\frac{n-1}{2}]$, where $[\ ]$ stands for the floor function (the function that maps a real number to the next smallest integer).*

6.9. In this subsection we explain two conjectural methods for finding $f^2_{sr}(n)$. Note that the question of decomposing the bimodules $\theta^l_{sr}(n)$ is discussed in [12, Section 4], from which it may be possible to deduce explicit formulas for the morphisms $f^2_{sr}(n)$.

To state the conjectures we have to introduce some morphisms. We start by recalling the Demazure operators $\partial_s : R \to R$, defined by $\partial_s(p) = \frac{p - s \cdot p}{2x_s}$. We can define the following morphisms of graded (R,R)-bimodules:

$$m_s : \theta_s \to R, \quad R \otimes_{R^s} R \ni p \otimes q \mapsto pq,$$
$$j_s : \theta_s\theta_s(2) \to \theta_s, \quad R \otimes_{R^s} R \otimes_{R^s} R \ni p \otimes q \otimes r \mapsto p\partial_s(q) \otimes r \in R \otimes_{R^s} R,$$
$$\alpha_s : R \to \theta_s\theta_s(2), \quad 1 \mapsto x_s \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x_s \in R \otimes_{R^s} R \otimes_{R^s} R.$$

It is an easy consequence of the construction of the light leaves basis (LLB) in [7] the fact that $f_{rs} \circ f_{sr}$ can be written as a linear combination of morphisms of the form $m_s, m_r, j_s, j_r, \alpha_s$ and α_r (eventually tensored, of course, by the identity). For example, if $m(s,r)=2$ then $f_{rs} \circ f_{sr} = \mathrm{id}$ and if $m(s,r)=3$ then

$$f_{rs} \circ f_{sr} = \mathrm{id} - \frac{1}{2\partial_s(x_r)}\big(\mathrm{id} \otimes m_r \otimes \mathrm{id}^2\big) \circ (\mathrm{id} \otimes \alpha_r \otimes \mathrm{id}) \circ (\mathrm{id} \otimes j_s) \circ (\alpha_s \otimes \mathrm{id}) \circ j_s \circ (\mathrm{id} \otimes m_r \otimes \mathrm{id}).$$

Conjecture 6.10. *Let us fix an integer $n \in \mathbb{N}$ and a Coxeter system $(W,\mathcal{S})$ with $s,r \in \mathcal{S}$ and $m(s,r)=n$. There exists a set $\{\lambda_i^{s,r}\}_{i\in I}$ of elements in the field of fractions of the integral domain $\mathbb{Q}[x,y]$ and another set $\{g_i^{s,r}\}_{i\in I}$ with each $g_i^{s,r}$ a composition of morphisms of the form $m_s, m_r, j_s, j_r, \alpha_s$ and α_r, eventually tensored by the identity, satisfying the equation $f_{rs} \circ f_{sr} = \sum_{i\in I} \lambda_i^{s,r}(\partial_s(x_r), \partial_r(x_s)) g_i^{s,r}$, for all Soergel's categories having W as the underlying group (this means that this formula is true for any reflection faithful representation V of W). For any Soergel's category having simple reflections s', r' with $n < m(s',r')$ we have the formula $f^2_{s'r'}(n) = \sum_{i\in I} \lambda_i^{s',r'}(\partial_{s'}(x_{r'}), \partial_r(x_{s'})) g_i^{s',r'}$.*

Remark 6.11. This conjecture is true for $n=2$ and $n=3$. In fact $f^2_{sr}(2) = \mathrm{id}$ and

$$f^2_{sr}(3) = \mathrm{id} - \frac{1}{2\partial_s(x_r)}\big(\mathrm{id} \otimes m_r \otimes \mathrm{id}^2\big) \circ (\mathrm{id} \otimes \alpha_r \otimes \mathrm{id}) \circ (\mathrm{id} \otimes j_s) \circ (\alpha_s \otimes \mathrm{id}) \circ j_s \circ (\mathrm{id} \otimes m_r \otimes \mathrm{id}).$$

We believe that $f^2_{sr}(n)$ is the composition of two maps $f_{sr}(n) \in A_n(s,r)$ and $f_{rs}(n) \in A_n(r,s)$, that are not bimodule morphisms, but only left R-module morphisms. This gives us our second conjectural method for finding $f^2_{sr}(n)$.

Conjecture 6.12. *Let us fix an integer $n \in \mathbb{N}$ and a Coxeter system $(W,\mathcal{S})$ with $s,r \in \mathcal{S}$ and $m(s,r)=n$. Let*

$$d = \prod_{\substack{t\in\mathcal{T} \\ t\leqslant sr(n)}} x_t. \tag{6.2}$$

There exist sets $\{\mu_i^{s,r,w}\}_{i\in I, w\in W}$ and $\{\nu_i^{s,r,w}\}_{j\in J, w\in W}$ of elements of the field of fractions of $\mathbb{Q}[x,y]$ and sets of polynomials in two variables $\{P_i^{s,r,w}\}_{i\in I, w\in W}$ and $\{Q_i^{s,r,w}\}_{j\in J, w\in W}$, satisfying the equations

$$\partial_w(d) = \sum_{i\in I} \mu_i^{s,r,w}\big(\partial_s(x_r), \partial_r(x_s)\big) P_i^{s,r,w}(x_s, x_r) \quad \textit{for all } w \in W, \tag{6.3}$$

$$\partial_w(d)^* = \sum_{j\in J} \nu_i^{s,r,w}\big(\partial_s(x_r), \partial_r(x_s)\big) Q_i^{s,r,w}(x_s, x_r) \quad \textit{for all } w \in W \tag{6.4}$$

for all Soergel's categories having W as the underlying group. For any Soergel's category having simple reflections s', r' with $n < m(s',r')$, if we repeat the formula of Proposition 3.6*,* *by replacing* d, $\partial_w(d)$ *and* $\partial_w(d)^*$ *as in* (6.2), (6.3) *and* (6.4), *we obtain a well-defined map* $f_{sr}(n) \in A_n(s,r)$, *that is not a bimodule morphism, but only a left* R*-module morphism. In the same way we can obtain* $f_{rs}(n) \in A_n(r,s)$. *We have the formula* $f_{sr}^2(n) = f_{rs} \circ f_{sr}$.

6.13. In this subsection we prove a generalization of Corollary 3.10.

Proposition 6.14. *Let* $s, r \in \mathcal{S}$ *and* $1 \leqslant n \leqslant m(s,r)$. *We have the following inclusion.*

$$f_{sr}^2(n)(1 \otimes_{R^s} \theta_r\theta_s \cdots) \subseteq 1 \otimes_{R^s} \theta_r\theta_s \cdots.$$

Proof. We will prove a more general result, namely that any degree zero morphism f in $\mathrm{End}(\theta_{sr}^l(n))$ satisfies the property

$$f(1 \otimes_{R^s} \theta_r\theta_s \cdots) \subseteq 1 \otimes_{R^s} \theta_r\theta_s \cdots \quad \big(\text{property } (**)\big)$$

We already know that this proposition is true if $n = m(s,r)$ (see 3.10), so from now on we will suppose $n < m(s,r)$. By Lemma 6.6 we have that zero is the minimal possible degree of a morphism in $\mathrm{End}(\theta_{sr}^l(n))$. So to prove property $(**)$ it is enough to prove it for the degree zero morphisms of a basis of the space $\mathrm{End}(\theta_{sr}^l(n))$ as left R-module.

From now on we will make heavy use of the notations and results in [7]. For $s \in \mathcal{S}$ we define the morphism

$$\alpha_s^i = \mathrm{id}^k \otimes \alpha_s \otimes \mathrm{id}^i \in \mathrm{Hom}(\theta_{r_1} \cdots \theta_{r_k}\theta_{t_1} \cdots \theta_{t_i}, \theta_{r_1} \cdots \theta_{r_k}\theta_s\theta_s\theta_{t_1} \cdots \theta_{t_i}).$$

A slight modification of the following lemma can be found in [7, Lemme 3.3], or with a more detailed proof in [9, Lemma 5.16].

Lemma 6.15. *For* $M, N \in \mathbf{B}$, *the morphism*

$$\begin{aligned} \mathrm{Hom}(M\theta_s, N) &\to \mathrm{Hom}(M, N\theta_s)(2) \\ f &\mapsto (f \otimes \mathrm{id}_{\theta_s}) \circ (\mathrm{id}_M \otimes \alpha_s) \end{aligned}$$

is an isomorphism of left graded R*-modules.*

Let $L = \{f_{\bar{i}}\}_{i \in I}$ be a light leaves basis (LLB) of $\mathrm{Hom}(\theta_{sr}^l(n)\theta_{sr}^r(n), R)$ as defined in [7, Théorème 5.1] (it is a basis as left R-module). We have that the sequence $(s_i)_i$ defined in [7, Section 4.5] is $(s_1, s_2, \ldots, s_n, s_{n+1}, \ldots, s_{2n}) = (s, r, s, \ldots, t, t, \ldots, s, r, s)$, where $t = r$ if n is even and $t = s$ if n is odd. We choose this LLB such that the $P(n, \bar{t})$ (see again [7, Section 4.5]) is a sequence of minimal length for all couples $(n, \bar{t})$. We will see in the sequel that there is only one LLB of $\mathrm{Hom}(\theta_{sr}^l(n)\theta_{sr}^r(n), R)$ satisfying the preceding property, and in fact all this sequences $P(n, \bar{t})$ are trivial (they do not play any role in defining L).

By Lemma 6.15 we deduce that the set

$$\big\{(f_{\bar{i}} \otimes \mathrm{id}_{\theta_{sr(n)}^l}) \circ \alpha_s^n \circ \alpha_r^{n-1} \circ \alpha_s^{n-2} \circ \cdots \circ \alpha_t^0\big\}_{i \in I}$$

is a basis of $\mathrm{End}(\theta_{sr}^l(n))$ as left R-module.

Given the form of α_s, our problem reduces to prove that for an element $f_{\bar{i}}$ of the LLB of $\mathrm{Hom}(\theta_{s_1}\cdots\theta_{s_{2n}}, R)$ of degree $-2n$, we have

$$f_{\bar{i}}(1 \otimes_{R^s} \theta_{s_2}\theta_{s_3}\cdots\theta_{s_{2n-1}} \otimes_{R^s} 1) \subseteq R^s. \tag{6.5}$$

We will follow the notation of [7, p. 17] with the only exceptions that the morphisms $m^s : \theta_s \to R$ and $i_1^s : \theta_s\theta_s \to \theta_s$ defined in [7, p. 12] will be called here m_s and j_s in concordance with the paper [9] and with Section 6.9. We will prove the following lemma. We put, $\bar{i} = (i_1, \ldots, i_{2n}) \in \{0,1\}^{2n}$, and

$$f_{\bar{i}} = f_{i_{2n},2n}^{j_{2n}} \circ \cdots \circ f_{i_1,1}^{j_1}(\mathrm{id}).$$

We denote $f_{\bar{i}}^k = f_{i_k,k}^{j_k} \circ \cdots \circ f_{i_1,1}^{j_1}(\mathrm{id})$.

If $M \in \mathbf{B}$ and $a \in \mathrm{Hom}(M, \theta_{t_1}\cdots\theta_{t_k})$ we denote $\varpi(a) = k$. If $M \in \mathbf{B}$ and $a \in \mathrm{Hom}(M, R)$, we denote $\varpi(a) = 0$.

Lemma 6.16.

(1) *$f_{\bar{i}}$ is a composition of morphisms of the form m_s, m_r, j_s and j_r tensored on both sides by the identity.*
(2) *For all $1 \leqslant k < 2n$, $\varpi(f_{\bar{i}}^k) \neq 0$.*

Proof. For all $0 \leqslant i, j \leqslant 1$, $1 \leqslant l \leqslant 2n$ and $a \in \mathrm{Hom}(\theta_{s_1}\cdots\theta_{s_{l-1}}, \theta_{t_1}\cdots\theta_{t_k})$ the definition of $f_{i,l}^j(a)$ gives the inequality

$$\left|\varpi\left(f_{i,l}^j(a)\right) - \varpi(a)\right| \leqslant 1. \tag{6.6}$$

So if (1) is not true, we have that for some $1 \leqslant k \leqslant 2n$, there is an f_{sr} or an f_{rs} tensored on both sides by the identity in the composition of morphisms defining $f_{i_k,k}^{j_k}$. This would mean that $\varpi(f_{\bar{i}}^{k-1}) = m(s,r)$, but this equation together with (6.6) gives that $k \geqslant m(s,r)$. Then $2n - k \leqslant 2n - m(s,r) < m(s,r)$, this last inequality because we have supposed $n < m(s,r)$. Finally $2n - k < m(s,r)$ and Eq. (6.6) contradict the fact that $\varpi(f_{\bar{i}}) = \varpi(f_{\bar{i}}^{2n}) = 0$. So we have proved (1).

Now we prove (2) by reduction to absurd. Let $1 \leqslant k < 2n$ be such that $\varpi(f_{\bar{i}}^k) = 0$. We assume that $1 \leqslant k \leqslant n$; the case $n \leqslant k \leqslant 2n$ has a similar proof. By the first part of the lemma we know that k must satisfy $s_k = s$, so k is odd, say $k = 2q+1$. By Proposition 6.8 we have that $\deg(f_{\bar{i}}^k) \geqslant -2q$.

By construction of the LLB, we have that $f_{\bar{i}} = f_{\bar{i}}^k \otimes g$, where g is an element of the LLB of $\mathrm{Hom}(\theta_{s_{k+1}}\cdots\theta_{s_{2n}}, R)$. By hypothesis, $\deg(f_{\bar{i}}) = -2n$, so $\deg(f_{\bar{i}}^k) + \deg(g) = -2n$, but because of Proposition 6.8 we know that

$$\begin{aligned}\deg(g) &\geqslant -2 - 2\left[\left(2n - (k+1) - 1\right)/2\right] \\ &= -2n + 2q + 2\end{aligned}$$

so $\deg(f_{\bar{i}}^k) + \deg(g) \geqslant -2q + (-2n + 2q + 2) = -2n + 2$, which is absurd and allows us to finish the proof of part (2). □

We can now finish the proof of the inclusion (6.5). Because of (1) and (2) we can see by induction in k that for $1 \leqslant k < 2n$ we have $f_{\bar{i}}^{k}(\theta_{s_1}\cdots\theta_{s_k}) \subseteq \theta_{t_1}\cdots\theta_{t_p}$ with $t_1 = s_1 = s$. So we have

$$\begin{aligned} f_{\bar{i}} &= f_{i_{2n},2n}^{j_{2n}} \circ f_{\bar{i}}^{2n-1} \\ &= (m_s \circ j_s) \circ \left(f_{\bar{i}}^{2n-1} \otimes \mathrm{id}_{\theta_s}\right). \end{aligned}$$

But because of (1) and of the explicit form of m_s, m_r, j_s and j_r given in Section 6.9, for $1 \leqslant k < 2n$ we have $f_{\bar{i}}^{k}(1 \otimes_{R^s} \theta_{s_2}\cdots\theta_{s_k}) \subseteq 1 \otimes_{R^s} \theta_{t_2}\cdots\theta_{t_p}$, so in particular $f_{\bar{i}}^{2n-1}(1 \otimes_{R^s} \theta_{s_2}\cdots\theta_{s_{2n-1}}) \subseteq 1 \otimes_{R^s} R \in \theta_s$. We conclude that

$$\begin{aligned} f_{\bar{i}}(1 \otimes_{R^s} \theta_{s_2}\cdots\theta_{s_{2n-1}} \otimes_{R^s} 1) &\subseteq (m_s \circ j_s)(1 \otimes_{R^s} R \otimes_{R^s} 1) \\ &= \partial_s(R) \\ &= R^s \end{aligned}$$

which proves the inclusion (6.5) and thus Proposition 6.14. □

6.17. *Definition and properties of D_w*

Let us generalize Definition 3.11.

Definition 6.18. When $i \in \mathbb{N}$ is such that $\mathrm{id}^i \otimes f_{s_{i+1},s_{i+2}}^2(k) \otimes \mathrm{id}^{n-i-k} \in \mathrm{Hom}(\theta_{s_1}\cdots\theta_{s_n}, \theta_{t_1}\cdots\theta_{t_n})$, then we denote by ${}^{i}f^2(k)$ the morphism $\mathrm{id}^i \otimes f_{s_{i+1},s_{i+2}}^2(k) \otimes \mathrm{id}^{n-i-k}$.

Notation 6.19. Even though there does not exist a bimodule morphism ${}^{i}f(k)$ we will use the notation ${}^{i}f^2(k) = {}^{i}f(k) \circ {}^{i}f(k)$.

Definition 6.20. A morphism g between $\theta_{s_1}\cdots\theta_{s_n}$ and $\theta_{t_1}\cdots\theta_{t_n}$ is of *Gf-type* (or generalized f-type) if there exists a tuple $(i_1,\ldots,i_k)$ such that

$$g = {}^{i_k}h_k \circ \cdots \circ {}^{i_2}h_2 \circ {}^{i_1}h_1,$$

where each h_j is either f or $f(k)$ for some integer k.

It is a (quite long but easy) exercise to repeat Section 3.9 and all Section 4 replacing everywhere cores by ELGcores (see Definition 4.2) and f-type by Gf-type. All definitions are straightforward in this context and all lemmas, propositions and the theorem have almost the same proofs. So we can define a special morphism of Gf-type, $\mathrm{G}f_{\bar{s}} = {}^{i_k}h_k \circ \cdots \circ {}^{i_2}h_2 \circ {}^{i_1}h_1 \in \mathrm{End}(\theta_{\bar{s}})$, characterized by the fact that, if $\bar{i} = (i_k,\ldots,i_1)$, then $N_{\bar{i}}(C) \neq 0$ for all $C \in \mathrm{ELGcores}(x)$.

As in Section 5 we can show that $\mathrm{G}f_{\bar{s}}$ exists and it is an idempotent uniquely defined by the preceding property. So if $\bar{s}$ is a reduced expression of w, we can define $D_{\bar{s}}$ as the image of $\mathrm{G}f_{\bar{s}}$ shifted by $l(w)$. In formulas this is $D_{\bar{s}} = \mathrm{G}f_{\bar{s}}(\theta_{\bar{s}})(l(w))$.

Theorem 6.21.

(1) *If $\bar{s}$ and $\bar{t}$ are two reduced expressions of $w \in W$ then $D_{\bar{s}}$ is isomorphic to $D_{\bar{t}}$. We call this (well-defined up to isomorphism) bimodule D_w.*
(2) *The indecomposable Soergel bimodule B_w is a direct summand of D_w and the set $\{\eta(\langle D_w\rangle)\}_{w\in W}$ is a basis of the Hecke algebra.*

Proof. The proof of part (1) is similar to that of Theorem 5.1 and the proof of part (2) is similar to that of Proposition 5.6. □

6.22. Some examples

(1) For all couples $(s,r) \in \mathcal{S}^2$ we have that $\eta(\langle D_x\rangle) = C'_x$ if $x \leqslant \underbrace{srs\cdots}_{m(s,r)\text{ terms}}$ and $l(x) \neq 3$.
(2) Let s, r, t, u, v be different elements of $\mathcal{S}$ such that $m(s,r) = 4$ and $m(s,t) = 4$. Then $\eta(\langle D_x\rangle) = C'_x$ if $x = srsrtst$, but $\eta(\langle D_x\rangle) \neq C'_x$ if $x = srsruvsrsr$.

We believe that D_w will categorify C'_w in a large range of cases but not in all cases (as the preceding examples show). It would be interesting to know exactly for what w this is the case.

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