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Presentation of right-angled Soergel categories by generators and relations

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ABSTRACT

We give a presentation (as a tensor category) by generators and relations of the category of Soergel's bimodules when the underlying group is a right-angled Coxeter group.

1. Introduction

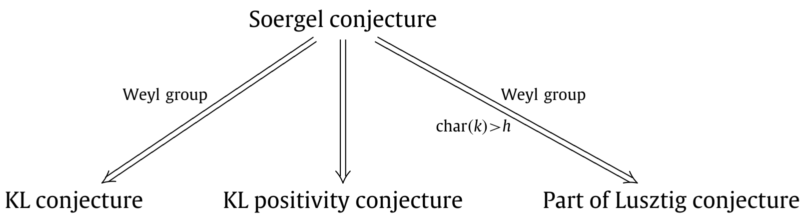
In 1979 [16], Kazhdan and Lusztig defined the Kazhdan–Lusztig polynomials, and this gave rise to what is now known as Kazhdan–Lusztig theory. They stated two major conjectures involving these polynomials in [16,17]: the Kazhdan–Lusztig conjecture (in the representation theory of complex semi-simple Lie algebras) and the Kazhdan–Lusztig positivity conjecture (in algebraic combinatorics). The first conjecture was proved for Weyl groups by Beilinson and Bernstein in [2] and by Brylinski and Kashiwara in [3] and later by Soergel [23] and Fiebig [9] using different approaches. The second conjecture was proved for Weyl or affine Weyl groups in [17] and in some other cases by Haddad [14] and Dyer [5].

In 1980, Lusztig stated a central conjecture in the representation theory, known as the Lusztig conjecture. This conjecture is about the characters of irreducible representations of reductive algebraic groups in positive characteristic. This conjecture is known only in large characteristics [1].

Let us consider $(W, \mathcal{S})$ a Coxeter system and $\mathcal{H}$ its Hecke algebra. In 1992, Soergel categorified $\mathcal{H}$ (see [24]). This means that he defined a tensor category $\mathbf{B}$ (that depends on a field k and on a representation of W) and an isomorphism of rings $\mathcal{E}$ from $\mathcal{H}$ to the split Grothendieck group of $\mathbf{B}$. He then stated a conjecture that links, via $\mathcal{E}$, the Kazhdan–Lusztig basis elements in $\mathcal{H}$ with the indecomposable elements of $\mathbf{B}$. This conjecture implies the Kazhdan–Lusztig positivity conjecture, and when the characteristic of k is larger that the Coxeter number h of W, it implies a part of the Lusztig conjecture. Moreover, Fiebig proves in [13] that a generalization of Soergel's conjecture for affine Coxeter systems implies the whole of Lusztig's conjecture.

Soergel introduced this category in the case of Weyl groups in order to make a link between the BGG category $\mathcal{O}$ of a semi-simple complex Lie algebra and semi-simple equivariant perverse sheaves on flag varieties. This link with geometry allowed him to prove his conjecture in the Weyl group case. He deduced in [23] the proof of the Kazhdan–Lusztig conjecture that we have mentioned.

The following diagram is a summary of the implications:



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In this paper, we are concerned with the case where $(W, \mathcal{S})$ is a right-angled Coxeter system. This means that $m(s, r) = 2$ or ∞ for all $s, r \in \mathcal{S}$. In this case, we find a presentation of the tensor category $\mathbf{B}$ (which we call the right-angled Soergel category) by generators and relations. If we can extend this result to Weyl and affine Weyl groups we expect to be able to re-prove the Kazhdan–Lusztig conjecture in the spirit of [23] but (from our perspective) in a more natural way. One should also be able to re-prove a part of Lusztig conjecture (comparing quantum and algebraic groups) in an essentially different way to the proof in [1].

Some months after this paper was submitted, Elias and Khovanov [8] found a presentation by generators and relations of Soergel's category when W is the symmetric group. Their proof relies strongly on diagrammatics and graph theory, in contrast to our method which is purely algebraic. Their result might help in the program of calculating Khovanov–Rozansky link homology (a categorification of the HOMFLYPT polynomial) defined in [18] using Soergel category $\mathbf{B}$. Some results in this direction are given in the papers [22,28]. We think that the approach in [8] is very interesting, but not easy to generalize to other cases. We believe that mixing their method and our method might yield the result in complete generality.

We want to remark that the right-angled case is very rich from a topological and geometrical perspective. For example, in the introduction of [4] Davis remarks that the right-angled case is sufficient for the construction of most examples of interest in geometric group theory.

This paper is divided as follows. In Section 2 we give the definition of Soergel's category and recall the *light leaves basis* of certain homomorphism spaces. In Section 3 we introduce the tensor category $\mathbf{T}$. This category is equivalent to the Soergel category of bimodules, and this will be proved in Section 4. In Section 4.1 we give a summary of the proof of this theorem.

2. Soergel's category of bimodules

We recall in this section the definition of Soergel's bimodules and the construction of a basis of the Hom spaces between some important Soergel bimodules in the right-angled case, as studied in [19].

2.1. Definition of Soergel bimodules

Let $(W, \mathcal{S})$ be a (not necessarily finite) Coxeter system with $|\mathcal{S}| < \infty$ and $\mathcal{T} \subset W$ the set of reflections in W, i.e.

$$\mathcal{T} = \bigcup_{w \in W} w\mathcal{S}w^{-1}.$$

Let k be an infinite field of characteristic different from 2 and V a finite dimensional k-representation of W. For $w \in W$, we denote by $V^w \subset V$ the set of w-fixed points.

In [26], Soergel defines a *reflection faithful representation* of $(W, \mathcal{S})$ as a faithful, finite dimensional representation V of W such that, for each $w \in W$, the subspace V^w is a hyperplane of V if and only if $w \in \mathcal{T}$.

From now on, we consider V a reflection faithful representation of W. If $k = \mathbb{R}$, by the results of [20], all the results in this paper will stay true if we consider V to be the geometric representation of W (even though this representation is not always reflection faithful).

Let $\widehat{R} = R(V)$ be the algebra of regular functions on V. We equip $\widehat{R}$ with a $\mathbb{Z}$-grading $R = \bigoplus_{i \in \mathbb{Z}} R_i$ such that we have $R_2 = V^*$ and $R_i = 0$ for uneven i. The action of W on V induces an action on $\widehat{R}$. For $s \in \mathcal{S}$ consider the $\mathbb{Z}$-graded $(\widehat{R}, \widehat{R})$-bimodule $\widehat{\theta}_s = \widehat{R} \otimes_{\widehat{R}^s} \widehat{R}$, where $\widehat{R}^s$ is the subspace of $\widehat{R}$ stabilized by s. The following is the central object of study in this paper:

Definition 2.1. Soergel's category $\mathbf{B}(W, V) = \mathbf{B}$ is the full subcategory of all $\mathbb{Z}$-graded $(\widehat{R}, \widehat{R})$-bimodules with objects the finite direct sums of direct summands of bimodules of the type $\widehat{\theta}_{s_1} \otimes_{\widehat{R}} \widehat{\theta}_{s_2} \otimes_{\widehat{R}} \cdots \otimes_{\widehat{R}} \widehat{\theta}_{s_n}$ for $(s_1, \ldots, s_n) \in \mathcal{S}^n$.

For simplicity, we will denote by $\widehat{\theta}_{s_1}\widehat{\theta}_{s_2}\cdots\widehat{\theta}_{s_n}$ the $(\widehat{R}, \widehat{R})$-bimodule

$$\widehat{\theta}_{s_1} \otimes_{\widehat{R}} \widehat{\theta}_{s_2} \otimes_{\widehat{R}} \cdots \otimes_{\widehat{R}} \widehat{\theta}_{s_n} \cong \widehat{R} \otimes_{\widehat{R}^{s_1}} \widehat{R} \otimes_{\widehat{R}^{s_2}} \cdots \otimes_{\widehat{R}^{s_n}} \widehat{R}.$$

More details about this category can be found in the papers by Dyer [6,7], by Fiebig [10–12], by myself [19,20], by Soergel [23–26] and by Williamson [27]. From our perspective, [26] forms the central and more complete reference.

We will use an auxiliary category in which we do not consider the direct summands nor the gradings: the category $\widehat{\mathbf{B}}$ is the category of $(\widehat{R}, \widehat{R})$-bimodules, with objects the finite direct sums of bimodules of the type $\widehat{\theta}_{s_1}\widehat{\theta}_{s_2}\cdots\widehat{\theta}_{s_n}$ for $(s_1, \ldots, s_n) \in \mathcal{S}^n$.

2.2. Some morphisms

In this subsection we will introduce some important morphisms between Soergel bimodules. Let $\widehat{x}_s \in V^*$ be an equation of the hyperplane H_s fixed by $s \in \mathcal{S}$. We have a decomposition $\widehat{R} \simeq \widehat{R}^s \oplus \widehat{x}_s\widehat{R}^s$, corresponding to

$$\widehat{R} \ni p = \frac{p + s \cdot p}{2} + \frac{p - s \cdot p}{2}.$$

We define $\widehat{P}_s(p) = (p + s \cdot p)/2$, $\widehat{I}_s(p) = (p - s \cdot p)/2$ and $\widehat{\partial}_s(p) = (p - s \cdot p)/2\widehat{x}_s$.

As W acts trivially over

$$\left(\sum_{s\in\mathcal{S}} k\widehat{x}_s\right)^{\perp} = \bigcap_{s\in\mathcal{S}} H_s,$$

we have that $V' = \sum_{s\in\mathcal{S}} k\widehat{x}_s$ is stabilized by W, so we deduce the important fact that $\widehat{P_t}(\widehat{x}_s), \widehat{I_t}(\widehat{x}_s) \in V'$.

We define four morphisms in $\mathbf{B}$ (or $\widehat{\mathbf{B}}$) that are the generators of Soergel's category as a tensor category. In other words, every morphism can be obtained as linear combinations of tensor products of compositions of these four morphisms together with identity morphisms, as is proved in [19]. The first is the multiplication morphism

$$\begin{array}{rcl} \widehat{m}_r : \widehat{\theta}_r & \to & \widehat{R} \\ \widehat{R}\otimes_{\widehat{R}^r}\widehat{R} \ni p_1\otimes p_2 & \mapsto & p_1p_2. \end{array}$$

The second is the only (up to non-zero scalar) degree -2 morphism from $\widehat{\theta}_r\widehat{\theta}_r$ to $\widehat{\theta}_r$:

$$\begin{array}{rcl} \widehat{j}_r : \widehat{\theta}_r\widehat{\theta}_r & \to & \widehat{\theta}_r \\ \widehat{R}\otimes_{\widehat{R}^r}\widehat{R}\otimes_{\widehat{R}^r}\widehat{R} \ni p_1\otimes p_2\otimes p_3 & \mapsto & p_1\widehat{\partial}_r(p_2)\otimes p_3. \end{array}$$

The third is the only (up to non-zero scalar) degree 2 morphism from $\widehat{R}$ to $\widehat{\theta}_r\widehat{\theta}_r$. This morphism is an important ingredient in some adjunctions that considerably reduce the complexity of the problem that we will attack in this paper:

$$\begin{array}{rcl} \widehat{\alpha}_r : \widehat{R} & \to & \widehat{\theta}_r\widehat{\theta}_r \\ 1 & \mapsto & \widehat{x}_r\otimes 1\otimes 1 + 1\otimes 1\otimes \widehat{x}_r. \end{array}$$

Finally, if $m(s,r) = 2$, we define the only (up to non-zero scalar) degree 0 morphism from $\widehat{\theta}_s\widehat{\theta}_r$ to $\widehat{\theta}_r\widehat{\theta}_s$. This morphism has an analogue when $m(s,r) > 2$ (see [21] for more details) but it is much more complicated to express it with a closed formula:

$$\begin{array}{rcl} \widehat{f}_{sr} : \widehat{\theta}_s\widehat{\theta}_r & \to & \widehat{\theta}_r\widehat{\theta}_s \\ \widehat{R}\otimes_{\widehat{R}^s}\widehat{R}\otimes_{\widehat{R}^r}\widehat{R} \ni p_1\otimes p_2\otimes p_3 & \mapsto & p_1\widehat{\partial}_s(p_2)\otimes 1\otimes \widehat{x}_s p_3 + p_1\widehat{P}_s(p_2)\otimes 1\otimes p_3. \end{array}$$

We define the following morphisms as compositions of the previous morphisms:

- $\widehat{p}_r = (\mathrm{id}\otimes\widehat{j}_r)\circ(\widehat{\alpha}_r\otimes\mathrm{id}) : \widehat{\theta}_r \to \widehat{\theta}_r\widehat{\theta}_r$;
- $\widehat{\epsilon}_r = (\mathrm{id}\otimes\widehat{m}_r)\circ\widehat{\alpha}_r : \widehat{R} \to \widehat{\theta}_r$;
- $\widehat{x}_r = (m_r\circ\epsilon_r)/2 : \widehat{R} \to \widehat{R}$.

It is easy to see that the morphism $\widehat{x}_r \in \mathrm{End}(\widehat{R})$ corresponds to the multiplication by $\widehat{x}_r \in \widehat{R}$.

2.3. A basis of the Hom spaces in the right-angled case

In the article [19], some bases are constructed (called BFL bases) for $\mathrm{Hom}_{(\widehat{R},\widehat{R})}(\widehat{\theta}_{s_1}\cdots\widehat{\theta}_{s_n}, \widehat{R})$ as a right $\widehat{R}$-module (with $(s_1,\ldots,s_n) \in \mathcal{S}^n$). We will fix one such basis by taking (with the notation of the paper [19]) $p(n, x, \bar{t})$ an m-tuple with m minimal. We have to introduce some notation in order to recall this basis.

If g is one of the morphisms defined in Section 2.2 then the morphism ${}^ig$ is, informally, the morphism g applied in the ith position tensored by the identity elsewhere. To be more precise, if they have a meaning, we define in $\mathrm{Hom}(\widehat{\theta}_{u_1}\cdots\widehat{\theta}_{u_l}, \widehat{\theta}_{t_1}\cdots\widehat{\theta}_{t_k})$ the following morphisms:

1. ${}^i\widehat{j} := \mathrm{id}^i\otimes\widehat{j}_{u_{i+1}}\otimes\mathrm{id}^{l-i-2}$.
2. ${}^i\widehat{m} := \mathrm{id}^i\otimes\widehat{m}_{u_{i+1}}\otimes\mathrm{id}^{l-i-1}$.
3. ${}^i\widehat{\alpha_s} := \mathrm{id}^i\otimes\widehat{\alpha}_s\otimes\mathrm{id}^{l-i}$.
4. ${}^i\widehat{f} := \mathrm{id}^i\otimes\widehat{f}_{u_{i+1}u_{i+2}}\otimes\mathrm{id}^{l-i-2}$.
5. ${}^i\widehat{p} := \mathrm{id}^i\otimes\widehat{p}_{u_{i+1}}\otimes\mathrm{id}^{l-i-1}$.
6. ${}^i\widehat{\epsilon_s} := \mathrm{id}^i\otimes\widehat{\epsilon}_s\otimes\mathrm{id}^{l-i-1}$.
7. ${}^i\widehat{x_s} := \mathrm{id}^i\otimes\widehat{x}_s\otimes\mathrm{id}^{l-i-1}$.

Let $\mathrm{Mo} = \{j, m, \alpha_s, f, p, \epsilon_s, x_s\}$. For $d \in \mathrm{Mo}$ we define $\widehat{d^i}$ in almost the same way, with the only difference that we put id^i in the right-hand side. For example $\widehat{j}^{\,i} = \mathrm{id}^{l-i-2}\otimes\widehat{j}_{u_{l-i-1}}\otimes\mathrm{id}^i$.

Now we are able to define the multiplication morphism $\widehat{m}(t,0)$, the chain morphism $\widehat{\mathrm{ch}}(t,t')$ and the complete chain morphism $\widehat{\mathrm{cch}}(t,t')$, for t, t' non-negative integers:

- $\widehat{m}(t,0) = \widehat{m}^t$.
- $\widehat{\mathrm{ch}}(t,0) = \widehat{j}^t$.

- $\widehat{\mathrm{ch}}(t, t' > 0) = \widehat{j}^t \circ \widehat{f}^{t+1} \circ \widehat{f}^{t+2} \circ \cdots \circ \widehat{f}^{t+t'}$.
- $\widehat{\mathrm{cch}}(t, t') = \widehat{m}^t \circ \widehat{\mathrm{ch}}(t, t')$.

Definition 2.2. Let $\mathrm{No} = \{\widehat{m}, \widehat{\mathrm{ch}}, \widehat{\mathrm{cch}}\}$. We say that $(\widehat{g_q}(t_q, t'_q), \ldots, \widehat{g_1}(t_1, t'_1))$, with $\widehat{g_i} \in \mathrm{No}$, is a good g-expression of the morphism $\widehat{g_q}(t_q, t'_q) \circ \cdots \circ \widehat{g_1}(t_1, t'_1)$ and that this expression has length q. If $t_{p+1} < t_p$ for all $1 \le p \le q$ then we say that the good g-expression is in the good order.

If $\nu = (\widehat{g_q}(t_q, t'_q), \ldots, \widehat{g_1}(t_1, t'_1))$ is a good g-expression of some morphism, $1 \le m \le q$ and $\widehat{g_m}(t_m, t'_m) \circ \cdots \circ \widehat{g_1}(t_1, t'_1) : \widehat{\theta}_{s_1} \cdots \widehat{\theta}_{s_n} \to \widehat{\theta}_{u_1} \cdots \widehat{\theta}_{u_l}$, we write $\mathrm{Codom}(\nu, m) = (u_1, \ldots, u_l)$. We also write $\mathrm{Codom}(\nu, 0) = (s_1, \ldots, s_n)$.

Definition 2.3. Let $\bar{t} = (t_1, \ldots, t_p) \in \mathcal{S}^p$. We say that the ith element of $\bar{t}$ is of left type if there exist $j < i$, with $t_j = t_i$ and $m(t_k, t_i) = 2$ for all $j < k < i$.

Let $\nu = (\widehat{g_q}(t_q, t'_q), \ldots, \widehat{g_1}(t_1, t'_1))$ be a good g-expression in the good order. Fix an integer $1 \le m \le q$ and write $\mathrm{Codom}(\nu, m-1) = (u_1, \ldots, u_l)$. If $(u_1, \ldots, u_{l-t_m-1})$ is a reduced expression in W we say that ν satisfies property (P_m). If ν satisfies property (P_m) for all $1 \le m \le q$, then we say that ν satisfies property (P).

Now we are able to define a basis of some Hom spaces. The following is a corollary of [19, thm. 5.1]:

Proposition 2.4. *Let $(W, \mathcal{S})$ be a right-angled Coxeter system. Let $(s_1, \ldots, s_n) \in \mathcal{S}^n$. The set*

$$\widehat{FL}(s_1, \ldots, s_n) = \{\text{morphisms in } \mathrm{Hom}(\widehat{\theta}_{s_1} \cdots \widehat{\theta}_{s_n}, \widehat{R}) \text{ associated with good } g\text{-expressions in good order satisfying property } (P)\}$$

is an $\widehat{R}$-basis of $\mathrm{Hom}_{(\widehat{R},\widehat{R})}(\widehat{\theta}_{s_1} \cdots \widehat{\theta}_{s_n}, \widehat{R})$ called the "light leaves basis".

We define two morphisms that will allow us to find many more homomorphism spaces than the ones given by the proposition. For every $M, N \in \widehat{\mathbf{B}}(W)$, we define the following two morphisms:

- $\widehat{F_s}(M, N): \mathrm{Hom}(\widehat{\theta}_s M, N) \to \mathrm{Hom}(M, \widehat{\theta}_s N)$, $f \mapsto (\mathrm{id}_{\widehat{\theta}_s} \otimes f) \circ (\widehat{\alpha}_s \otimes \mathrm{id}_M)$.
- $\widehat{G_s}(M, N): \mathrm{Hom}(M, \widehat{\theta}_s N) \to \mathrm{Hom}(\widehat{\theta}_s M, N)$, $g \mapsto ((\widehat{m}_s \circ \widehat{j_s}) \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\widehat{\theta}_s} \otimes g)$.

We will write $\widehat{F_s}$ and $\widehat{G_s}$ when M and N are known. The morphisms $\widehat{F_s}(M, N)$ and $\widehat{G_s}(M, N)$ are inverse to each other (see [19, lemma 3.3] and its proof), so, for every couple of sequences $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_k)$, the set

$$\widehat{FL}(s_1, \ldots, s_n; t_1, \ldots, t_k) := \widehat{F_{t_1}} \circ \cdots \circ \widehat{F_{t_{k-1}}} \circ \widehat{F_{t_k}} \circ \widehat{FL}(s_1, \ldots, s_n)$$

is a basis of $\mathrm{Hom}(\widehat{\theta}_{s_1} \cdots \widehat{\theta}_{s_n}, \widehat{\theta}_{t_1} \cdots \widehat{\theta}_{t_k})$.

3. The tensor category T

The main goal of this section is to introduce the tensor category **T**. We will see in the sequel that this category (more precisely, a slight variation of it) is equivalent to Soergel's category in the right-angled case (see Theorem 3.3 for more details). We will start by defining in 3.1 the tensor category T_r that is equivalent to the category **T** in rank 1. We will introduce in 3.2 the tensor category **T** by generators and relations.

3.1. The tensor category T_r

For an introduction to tensor categories and strict tensor categories with a presentation by generators and relations, we refer the reader to [15, chapters XI and XII].

We define the strict tensor category T_r by generators and relations. Its objects are generated by θ_r and by the unit R as tensor category. Its morphisms are generated by:

- $j_r : \theta_r \to \theta_r\theta_r$.
- $m_r : \theta_r \to R$.
- $\alpha_r : R \to \theta_r\theta_r$.

We define p_r, ϵ_r and x_r by the same formulas as in Section 2.2 (omitting all hats). The relations defining T_r are the following:

1. $\epsilon_r = (m_r \otimes \mathrm{id}) \circ \alpha_r : R \to \theta_r$.
2. $p_r = (j_r \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha_r) : \theta_r \to \theta_r\theta_r$.
3. $(\mathrm{id} \otimes (m_r \circ j_r)) \circ (\alpha_r \otimes \mathrm{id}) = \mathrm{id} : \theta_r \to \theta_r$.
4. $((m_r \circ j_r) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha_r) = \mathrm{id} : \theta_r \to \theta_r$.
5. $j_r \circ (\mathrm{id} \otimes j_r) = j_r \circ (j_r \otimes \mathrm{id}) : \theta_r\theta_r\theta_r \to \theta_r$.
6. $j_r \circ \alpha_r = 0 : R \to R$.

7. $\mathrm{id} \otimes m_r = m_r \otimes \mathrm{id} + j_r x_r - x_r j_r : \theta_r\theta_r \to \theta_r$.
8. $j_r \circ (\mathrm{id} \otimes x_r \otimes \mathrm{id}) = (m_r \otimes \mathrm{id}) - (x_r \otimes \mathrm{id}) \circ j_r : \theta_r\theta_r \to \theta_r$.

The following definition is needed to state Proposition 3.2.

Definition 3.1. Let A be a commutative ring and $\mathcal{C}$ an A-linear category. If R is a commutative A-algebra, we define the category $\mathcal{C} \otimes_A R$ in the following way: it has the same objects as $\mathcal{C}$ and its morphisms are defined by the formula

$$\mathrm{Hom}_{\mathcal{C}\otimes_A R}(M, N) = \mathrm{Hom}_{\mathcal{C}}(M, N) \otimes_A R.$$

Let $\mathcal{A}$ be the subring of $\mathrm{End}_{T_r}(R)$ generated by the set $\{x_s\}_{s\in\mathcal{S}}$. The following proposition is a special case of Theorem 3.3 that will be proved in the sequel.

Proposition 3.2. *Let W be a Weyl group of type A_1. The application $x_s \mapsto \widehat{x}_s$ extends to a morphism $\mathcal{A} \to \widehat{R}$ and there is an equivalence of $\widehat{R}$-linear tensor categories between $\widehat{\mathbf{B}}(W)$ and $T_r \otimes_{\mathcal{A}} \widehat{R}$.*

3.2. Definition of $\mathbf{T}$

Now we are ready to define the tensor category $\mathbf{T}$ by generators and relations. Let $(W, \mathcal{S})$ be a right-angled Coxeter system and V a reflection faithful representation of W. We will define the tensor category $\mathbf{T}(W, V)$ by generators and relations. Its objects are generated as a tensor category by θ_r for all $r \in \mathcal{S}$, and by the unit R. Its morphisms are generated by

- $j_r : \theta_r\theta_r \to \theta_r$.
- $m_r : \theta_r \to R$.
- $\alpha_r : R \to \theta_r\theta_r$.
- $f_{sr} : \theta_s\theta_r \to \theta_r\theta_s$.

We define p_r, ϵ_r and x_r by the same formulas as in Section 2.2 without the hats. For $s, r \in \mathcal{S}$, let us write (see Section 2.2)

$$\widehat{P}_t(\widehat{x}_s) = \sum_{r\in\mathcal{S}} \lambda^r_{t,s}\widehat{x}_r, \qquad \widehat{I}_t(\widehat{x}_s) = \mu_{t,s}\widehat{x}_t \quad \text{and} \quad \widehat{\partial}_t(\widehat{x}_s) = \frac{\widehat{I}_t(\widehat{x}_s)}{\widehat{x}_t},$$

with $\lambda^r_{t,s}$ and $\mu_{t,s}$ elements of the field k. Using these coefficients we define

$$P_t(x_s) = \sum_{r\in\mathcal{S}} \lambda^r_{t,s}x_r, \qquad I_t(x_s) = \mu_{t,s}x_t \quad \text{and} \quad \partial_t(x_s) = \frac{I_t(x_s)}{x_t}.$$

We define a monomial in $\mathrm{End}(R)$ to be a scalar multiple of a composition of elements of the form x_s. Similarly a polynomial $\lambda \in \mathrm{End}(R)$ is a linear combination of monomials with coefficients in k. We will write $\mathcal{A} \subseteq \mathrm{End}(R)$ for the subring of polynomials.

The relations defining $\mathbf{T}$ include the first seven relations in the definition of T_r (Section 3.1) and the following new relations (from **a** to **e** we assume that $m(s, r) = m(r, t) = m(s, t) = 2$):

(**a**) $f_{sr} \circ f_{rs} = \mathrm{id} : \theta_s\theta_r \to \theta_s\theta_r$.
(**b**) $(m_r \otimes \mathrm{id}) \circ f_{sr} = \mathrm{id} \otimes m_r : \theta_s\theta_r \to \theta_s$.
(**c**) $(j_s \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f_{rs}) = f_{rs} \circ (\mathrm{id} \otimes j_s) \circ (f_{sr} \otimes \mathrm{id}) : \theta_s\theta_r\theta_s \to \theta_s\theta_r$.
(**d**) $(\mathrm{id} \otimes f_{sr}) \circ (\alpha_s \otimes \mathrm{id}) = (f_{rs} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha_s) : \theta_r \to \theta_s\theta_r\theta_s$.
(**e**) $(\mathrm{id} \otimes f_{sr}) \circ (f_{st} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f_{rt}) = (f_{rt} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f_{st}) \circ (f_{sr} \otimes \mathrm{id}) : \theta_s\theta_r\theta_t \to \theta_t\theta_r\theta_s$.
(**f**) $f_{tr} \circ (\mathrm{id}_{\theta_r} \otimes x_s \otimes \mathrm{id}_{\theta_r}) = (P_t(x_s) \otimes \mathrm{id}_{\theta_r\theta_t}) \circ f_{tr} + (\mathrm{id}_{\theta_r\theta_t} \otimes I_t(x_s)) \circ f_{tr} : \theta_t\theta_r \to \theta_r\theta_t$.
(**g**) $j_r \circ (\mathrm{id}_{\theta_r} \otimes x_s \otimes \mathrm{id}_{\theta_r}) = \partial_r(x_s)m_r \otimes \mathrm{id} + (P_r(x_s) \otimes \mathrm{id} - x_s\partial_r(x_s) \otimes \mathrm{id}) \circ j_r : \theta_r\theta_r \to \theta_r$.

Heuristically **a**, **b**, **c** and **d** tell us that the f_{sr} "commute" with everything. The relation **e** is the hexagon relation typical in the symmetric monoidal categories. The last two relations tells us how to "take out" x_s.

The ring $\mathrm{End}(R)$ is commutative because of general results in tensor categories (see [15, prop. X1.2.4]). As before, for $s \in \mathcal{S}$, we define the action of $x_s \in \mathrm{Arr}(\mathbf{T}(W, V))$ in $\widehat{R}$ in the same way as the action of $\widehat{x}_s$ in $\widehat{R}$. We deduce an action of $\mathcal{A}$ in $\widehat{R}$.

The following is the central result of this paper:

Theorem 3.3. *Let $(W, \mathcal{S})$ be a right-angled Coxeter group. The functor that sends θ_s to $\widehat{\theta}_s$, $j_s \otimes 1$ to $\widehat{j}_s$, etc. is an equivalence of $\widehat{R}$-linear tensor categories between $\mathbf{T}(W, V) \otimes_{\mathcal{A}} \widehat{R}$ and $\widehat{\mathbf{B}}(W, V)$.*

In the next section we will prove this theorem.

4. Proof of the main theorem

The goal of this section is to prove Theorem 3.3.

4.1. Summary of the proof

Before we can give a summary of the proof of Theorem 3.3 we have to introduce some more notation. A *basic bimodule* is a bimodule of the form $\theta_{s_1} \cdots \theta_{s_n}$, with $(s_1, \ldots, s_n) \in \mathcal{S}^n$. Every morphism between basic bimodules can be written as a sum of elements of the following type: ${}^{i_\omega}d_\omega \circ \cdots \circ^{i_1} d_1$ (see Section 2.3), with

$$d_i \in \mathrm{Lo} := \{j_s, m_s, f_{sr}, \alpha_s \mid s, r \in \mathcal{S}\}.$$

An *expression* of a morphism g between basic bimodules is a sequence $({}^{i_\omega}d_\omega, \ldots, {}^{i_1}d_1)$, with $d_i \in \mathrm{Lo}$, such that ${}^{i_\omega}d_\omega \circ \cdots \circ^{i_1} d_1 = g$. We remark the fact that in general there are many expressions for each morphism.

Sometimes we will only say that ν is an expression (without specifying an expression of what morphism). We will also say that the expression ν *represents* the morphism g and we write $\overline{\nu} = g$. An *R-expression* is an expression of a morphism with domain a basic bimodule and codomain R. Sometimes we will consider a good g-expression simply like an expression in the obvious way.

We need to consider an important set of R-expressions: G is the set of all scalar multiples of R-expressions $({}^{i_\omega}d_\omega, \ldots, {}^{i_1}d_1)$ with $d_i \neq \alpha_s$ for all $1 \leq i \leq \omega$ and for all $s \in \mathcal{S}$.

In Section 4.3 we will define the two integers $\nu\{\#m\text{-bad}\}$ and $\nu\{\#j\text{-bad}\}$ associated with an expression $\nu \in G$ that measure how far $\overline{\nu}$ is from being in the light leaves basis: we will see that if ν is an expression of $\overline{\nu}$, then $\overline{\nu}$ is in $FL(s_1, \ldots, s_n)$ if and only if ν is a good g-expression in the good order, and $\nu\{\#m\text{-bad}\} = \nu\{\#j\text{-bad}\} = 0$. So all of the proof will consist in coming closer and closer to this description.

In Section 4.2 we will reduce the problem to considering only R-expressions. As we will apply many algorithms we will need some numbers associated with each expression that will ensure that the algorithms have an end, that they cannot be applied indefinitely. These numbers are introduced in Section 4.3.

In each one of the following Sections 4.4–4.9 we will start with an R-expression satisfying some properties and then we will find polynomials λ_i and expressions ν_i such that

$$\overline{\nu} = \sum_i \lambda_i \overline{\nu_i},$$

with all the ν_i satisfying some new properties. In the following list, we show what properties satisfy the R-expression with which we start in each section, and what properties satisfy the expressions at which we arrive:

- Section 4.4: $\nu \in G \rightsquigarrow \nu \in G,\ \nu\{\#m\text{-bad}\} = 0$.
- Section 4.5: $\nu \in G \rightsquigarrow \nu \in G,\ \nu\{\#m\text{-bad}\} = 0$, ν a good g-expression.
- Section 4.6: $\nu \rightsquigarrow \nu \in G$.
- Section 4.7: ν a good g-expression $\rightsquigarrow \nu$ a good g-expression in the good order.
- Section 4.8: $\nu \in G \rightsquigarrow \nu$ a good g-expression in the good order with $\nu\{\#m\text{-bad}\} = 0$.
- Section 4.9: ν a good g-expression in the good order with $\nu\{\#m\text{-bad}\} = 0 \rightsquigarrow \nu$ a good g-expression in the good order with $\nu\{\#m\text{-bad}\} = \nu\{\#j\text{-bad}\} = 0$.

4.2. Reduction of the problem to R-expressions

We will start by defining a functor from $\mathbf{T} := \mathbf{T}(W, V) \otimes_{\mathcal{A}} \widehat{R}$ to $\widehat{\mathbf{B}} := \widehat{\mathbf{B}}(W, V)$. To this end we repeat all the definitions in Section 2.3 taking out the hats: for $d \in \mathrm{Mo}$ we define ${}^i d$ and d^i, we define $m(t, t')$, $\mathrm{ch}(t, t')$, $\mathrm{cch}(t, t')$, a good g-expression, to be of left type, FL, $F_s(M, N)$ and $G_s(M, N)$.

To define the functor $\mathfrak{Fu}$ from $\mathbf{T}$ to $\widehat{\mathbf{B}}$ we have to define it in the objects and in the morphisms. We define $\mathfrak{Fu}(R) = \widehat{R}$, and for all $s \in \mathcal{S}$, $\mathfrak{Fu}(\theta_s) = \widehat{\theta_s}$. If M and M' are objects of $\mathbf{T}$, we define $\mathfrak{Fu}(M \otimes M') = \mathfrak{Fu}(M) \otimes \mathfrak{Fu}(M')$. For all $s \in \mathcal{S}$ we define $\mathfrak{Fu}(j_s \otimes \lambda) = \widehat{j_s}\lambda$, $\mathfrak{Fu}(m_s \otimes \lambda) = \widehat{m}_s\lambda$, etc.

As we know explicitly all the morphisms in $\widehat{\mathbf{B}}$, we can easily verify that all the relations are satisfied in $\widehat{\mathbf{B}}$, so by [15, proposition XII.1.4] we have that $\mathfrak{Fu}$ defines a tensor functor. By [19, thm. 5.1] we know that the set of morphisms $\{\widehat{j}_r, \widehat{m}_r, \widehat{\alpha}_r, \widehat{f}_{sr}\}$ generate (as a tensor category) all the morphisms in $\widehat{\mathbf{B}}$. So we only need to prove that for all $M, N \in \mathbf{T}$, the map $\mathfrak{Fu} : \mathrm{Hom}_{\mathbf{T}}(M, N) \to \mathrm{Hom}_{\widehat{\mathbf{B}}}(\mathfrak{Fu}(M), \mathfrak{Fu}(N))$ is injective.

We start by proving the following:

Lemma 4.1. *The applications $F_s(M, N)$ and $G_s(M, N)$ are inverse to each other in* $\mathbf{T}$.

Proof.

$$\begin{aligned}
F_s(M,N) \circ G_s(M,N)(g) &= F_s(M,N)(((m_s \circ j_s) \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\theta_s} \otimes g)) \\
&= \left\{\mathrm{id}_{\theta_s} \otimes \left[((m_s \circ j_s) \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\theta_s} \otimes g)\right]\right\} \circ (\alpha_s \otimes \mathrm{id}_M) \\
&= \left[(\mathrm{id}_{\theta_s} \otimes (m_s \circ j_s) \otimes \mathrm{id}_N) \circ (\mathrm{id}^2_{\theta_s} \otimes g)\right] \circ (\alpha_s \otimes \mathrm{id}_M) \\
&= (\mathrm{id}_{\theta_s} \otimes (m_s \circ j_s) \otimes \mathrm{id}_N) \circ (\alpha_s \otimes g) \\
&= (\mathrm{id}_{\theta_s} \otimes (m_s \circ j_s) \otimes \mathrm{id}_N) \circ (\alpha_s \otimes \mathrm{id}_{\theta_s} \otimes \mathrm{id}_N) \circ g \\
&= \left(\left\{\left[\mathrm{id}_{\theta_s} \otimes (m_s \circ j_s)\right] \circ (\alpha_s \otimes \mathrm{id}_{\theta_s})\right\} \otimes \mathrm{id}_N\right) \circ g \\
&= (\mathrm{id}_{\theta_s} \otimes \mathrm{id}_N) \circ g \\
&= g.
\end{aligned}$$

If (0) is the relation $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (f_1 \circ f_2) \otimes (g_1 \circ g_2)$ (a relation satisfied in all tensor categories), then all but the next to last equality are derived from (0). The next to last equality is derived from relation 3 (see the definition of T_r in Section 3.1). In a similar way we prove that $G_s(M, N) \circ F_s(M, N)(f) = f$ using relation 4. □

The following lemma allows us to reduce the problem to the R-expressions.

Lemma 4.2. *Let us suppose that for every sequence* $(s_1, \ldots, s_n) \in \mathcal{S}^n$, *every element* $f \in \mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, R)$ *can be written in the form* $f = \sum_i a_i\lambda_i$, *where* $a_i \in FL(s_1, \ldots, s_n)$ *and the* λ_i *are polynomials. Then for every couple of sequences* $(s_1, \ldots, s_n)$ *and* $(t_1, \ldots, t_k)$, *every element* $f' \in \mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, \theta_{t_1} \cdots \theta_{t_k})$ *can be written in the form* $f' = \sum_i a'_i\lambda'_i$, *where* $a'_i \in FL(s_1, \ldots, s_n; t_1 \cdots t_k)$ *and the* λ'_i *are polynomials.*

Proof. By the hypothesis, $G_{t_k} \circ \cdots \circ G_{t_1}(f') = \sum_i a_i\lambda_i$, where $a_i \in FL(t_k, \ldots, t_1, s_1, \ldots, s_n)$ and the λ_i are polynomials. Then

$$\begin{aligned} f' &= F_{t_1} \circ \cdots \circ F_{t_k} \circ G_{t_k} \circ \cdots \circ G_{t_1}(f') \\ &= \textstyle\sum_i F_{t_1} \circ \cdots \circ F_{t_k}(a_i\lambda_i) \\ &= \textstyle\sum_i \left\{F_{t_1} \circ \cdots \circ F_{t_k}(a_i)\right\} \lambda_i \end{aligned}$$

and by definition $F_{t_1} \circ \cdots \circ F_{t_k}(a_i) \in FL(s_1, \ldots, s_n; t_1 \cdots t_k)$ for all i. □

From this lemma we can conclude that to complete the proof of Theorem 3.3, it suffices to prove that for every sequence $(s_1, \ldots, s_n)$, if $f \in \mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, R)$, then f can be written in the form $\sum_i a_i\lambda_i$, where $a_i \in FL(s_1, \ldots, s_n)$ and the λ_i are polynomials.

Before we finish this section we will introduce some notation that will formalize some natural concepts such as *applying* a relation to an expression. If $({}^{i_\omega}d_\omega, \ldots, {}^{i_1}d_1)$ is an expression of g, then ${}^{i_k}d_k$ is the kth term, ${}^{i_\omega}d_\omega$ is the last term and ${}^{i_1}d_1$ is the first term of this expression. By abuse of notation, sometimes when this leads to no confusion, we will write d instead of ${}^{i}d$, x instead of x_s, f instead of f_{sr}, etc.

We will say that $\nu = ({}^{i_\omega}h_\omega, \ldots, {}^{i_1}h_1)$ with $h_i \in \mathrm{Mo}$ (see Section 2.3) for all $1 \leq i \leq \omega$ is a generalized expression of ${}^{i_\omega}h_\omega \circ \cdots \circ {}^{i_1}h_1$. We name as $\mathcal{LCE}$ the set of formal $\mathcal{A}$-linear combinations of generalized expressions. If

$$\nu = \sum_{i \in I} \lambda_i \nu_i \in \mathcal{LCE},$$

with $\lambda_i \in \mathcal{A}$, we define

$$\overline{\nu} = \sum_{i \in I} \lambda_i \overline{\nu_i}.$$

Let the following equation:

$${}^{i_\omega}h_\omega \circ \cdots \circ {}^{i_1}h_1 = \sum_z \lambda_z \, {}^{p_{u,z}}h'_{u,z} \circ \cdots \circ {}^{p_{1,z}}h'_{1,z}$$

represent the relation X (any one of the relations 1–8 or a–g defining the tensorial category $\mathbf{T}$), where all h_i and $h'_{i,z}$ are elements of Mo, and λ_z are polynomials. For all natural numbers $b \geq 0$ we say that $({}^{i_\omega+b}h_\omega, \ldots, {}^{i_1+b}h_1)$ is a left X-relation and $\sum_z(\lambda_z, {}^{p_{u,z}+b}h'_{u,z}, \ldots, {}^{p_{1,z}+b}h'_{1,z})$ is the corresponding right X-relation.

If ν is a generalized expression, then we say that we *apply* the relation X to ν, and we obtain a linear combination $\sum_z \lambda_z \nu'_z$ of generalized expressions if $\sum_z \lambda_z \nu'_z$ is the result of changing, in ν, a substring that is a left X-relation to the corresponding right X-relation.

4.3. Some tuples of integers associated with an expression

In this section we will introduce some applications from the set of R-expressions to $\mathbb{N}^p$ for some natural numbers p. These applications may seem strange at first, because they are only technical instruments for proving that some algorithms converge, but they will appear more natural in the context of the proof.

Let $\nu = ({}^{i_k}d_k, \ldots, {}^{i_1}d_1)$ be an expression; we will write $\nu_r = ({}^{i_r}d_r, \ldots, {}^{i_1}d_1)$, the truncation of ν at r. We define the sequence of integers $(i'_1, i'_2, \ldots, i'_k)$ such that $\overline{\nu} = d_k^{i'_k} \circ \cdots \circ d_1^{i'_1}$. If $\overline{\nu_r} : \theta_{s_1} \cdots \theta_{s_n} \to \theta_{t_1} \cdots \theta_{t_p}$, we define $\mathrm{Codom}(\nu_r) = (t_1, \ldots, t_p) \in \mathcal{S}^p$. For $\delta \in \{j, m, \alpha, f\}$ we define

$$\nu[\delta] = \{p \mid 1 \leq p \leq k \text{ and } d_p = \delta\}.$$

We say that the integer r is m-bad (resp. j-bad) for ν if $d_r = m$ (resp. $d_r = j$) and the $(i_r + 1)$th element of $\mathrm{Codom}(\nu_r)$ is of left type (recall Definition 2.3).

We define the following sets associated with ν:

- $A_m(\nu) = \{r \mid r \text{ is } m\text{-bad for } \nu\}$.
- $A_j(\nu) = \{r \mid r \text{ is } j\text{-bad for } \nu\}$.

The following two applications will be fundamental in the sequel:

- $\nu\{\#m\text{-bad}\} = \mathrm{card}(A_m(\nu)) \in \mathbb{N}$.
- $\nu\{\#j\text{-bad}\} = \mathrm{card}(A_j(\nu)) \in \mathbb{N}$.

We advise the reader that it is not necessary to read the following notation immediately. It may be more useful to come back to this section every time these applications are needed.

Let $\nu[j] = \{a_1, \ldots, a_p\}$ with $a_1 < a_2 < \cdots < a_p$.

- $\nu\{\text{positions of } j\text{'s}\} = (a_1 + i'_{a_1}, \ldots, a_p + i'_{a_p}) \in \mathbb{N}^p$.
- $\nu\{\#f\} = \mathrm{card}(\nu[f]) \in \mathbb{N}$.
- $\nu\{\#m, j\} = \mathrm{card}(\nu[m] \cup \nu[j]) \in \mathbb{N}$.
- $\nu\{f \text{ to the right}\} = \sum_{p\in\nu[f]}(i_p) \in \mathbb{N}$.
- $\nu\{\text{depth } m \text{ and } j\} = \sum_{p\in\nu[m]\cup\nu[j]}(p) \in \mathbb{N}$.
- $\nu\{m \text{ far from bottom}\} = \sum_{p\in\nu[m]}(k-p) \in \mathbb{N}$.
- $\nu\{\min m\text{-bad}\} = \begin{cases} \min A_m(\nu) & \text{if } A_m(\nu) \neq \emptyset \\ 0 & \text{if } A_m(\nu) = \emptyset \end{cases}.$
- $\nu\{\#m, j \text{ after min. } m\text{-bad}\} = \mathrm{card}(\{p > \nu\{\min m\text{-bad}\} \mid d_p = m \text{ or } d_p = j\}) \in \mathbb{N}$.
- $\nu\{\text{function of } m\text{-bad}\} = (\nu\{\#m, j \text{ after min. } m\text{-bad}\}, i_{\nu\{\min m\text{-bad}\}}) \in \mathbb{N}^2$.
- $\nu\{\max. j\text{-bad}\} = \begin{cases} \max A_j(\nu) & \text{if } A_j(\nu) \neq \emptyset \\ 0 & \text{if } A_j(\nu) = \emptyset. \end{cases}$

If $p \in \nu[m]$ (resp. $p \in \nu[j]$) and $\mathrm{Codom}(\nu_p) = (t_1, \ldots, t_k)$, then we define $\pi_m(p)$ (resp. $\pi_j(p)) = \mathrm{card}\{a \le i_p \mid t_a = t_{i_{p+1}}\}$. Finally we define

$$\nu\{m, j \text{ equal to left}\} = \sum_{p\in\nu[m]\cup\nu[j]} (\pi_m(p) + \pi_j(p)) \in \mathbb{N}.$$

4.4. *Restriction to the case $\nu\{\#m\text{-bad}\} = 0$*

The following proposition is the first step in the project described in the last part of Section 4.1.

Proposition 4.3. *For every $\nu \in G$ (see Section 4.1) there exist a set Δ, polynomials λ_δ and elements $\nu_\delta \in G$ such that $\overline{\nu} = \sum_{\delta\in\Delta} \lambda_\delta \nu_\delta$ and such that $\nu_\delta\{\#m\text{-bad}\} = 0$ for all $\delta \in \Delta$.*

Proof. We will construct $\mathcal{F}_1(\nu)$ and $\mathcal{F}_2(\nu)$, linear combinations of expressions such that $\overline{\mathcal{F}_i(\nu)} = \overline{\nu}$ for $1 \le i \le 2$.

4.4.1. *$\mathcal{F}_1(\nu)$: "Taking out the x_s"*

In this section we prove that we can "take out" an x_s that is "inside" an element of G; we need a definition to explain this in a rigorous way in Lemma 4.5.

Definition 4.4. We define G^p_η as the set of all scalar multiples of generalized expressions of the form $\nu = ({}^{i_p}d_p, \ldots, {}^{i_{\eta+1}}d_{\eta+1}, {}^{i}x_s, {}^{i_\eta}d_\eta, \ldots, {}^{i_1}d_1)$, with $d_l \in \mathrm{Lo} - \{\alpha_s\}_{s\in\mathcal{S}}$ for all $1 \le l \le p$. We define $G^p_{\ge\eta} = \bigcup_{\eta'\ge\eta} G^p_{\eta'}$.

Lemma 4.5. *Let $\nu \in G^p_\eta$. There exist an element $\mathcal{F}_1(\nu)$, polynomials λ_i of degree ≤ 1 and elements $g_i \in G$ such that $\mathcal{F}_1(\nu) = \sum_{i\in I} \lambda_i g_i$ and $\overline{\mathcal{F}_1(\nu)} = \overline{\nu}$.*

Proof. If $g : M_1 \to N_1$ and $f : M_2 \to N_2$ are two morphisms, we call the relation $(f\otimes \mathrm{id}_{N_1})\circ(\mathrm{id}_{M_2}\otimes g) = (\mathrm{id}_{N_2}\otimes g)\circ(f\otimes \mathrm{id}_{m_1})$, satisfied in all tensor categories, a *commutation relation*.

Let $\nu \in G^p_\eta$. Let us take x_s as far to the left as possible with commutation relations. By this we mean that we apply commutation relations to ν so as to find a new expression of $\overline{\nu}$ that belongs to $G^p_{\eta'}$ with η' maximal. If x_s does not arrive at the last term (i.e. $\eta' \neq p$), we obtain a substring of the form $({}^{i-1}j_r, {}^{i}x_s)$ or of the form $({}^{i-1}f_{tr}, {}^{i}x_s)$. If we are in the first case we apply relation $\mathbf{g}$ and we obtain an element of G plus a finite sum of elements in $G^p_{\eta'+1}$. If we are in the second case, we apply relation $\mathbf{f}$ and we obtain a finite sum of elements in $G^p_{\eta'+1}$. If we repeat this procedure enough times we arrive at the desired result, but let us be more precise. We will write $\mathcal{F}_1^1(\nu)$ for the element obtained:

$$\mathcal{F}_1^1(\nu) = \sum_{i\in I_1} w_i^1 + \sum_{b\in B_1} g_b^1.$$

Here $\overline{\nu} = \overline{\mathcal{F}_1^1(\nu)}$, $g_b^1 \in G$ (the cardinal of B_1 is 1 if we applied relation $\mathbf{g}$ and 0 if we applied relation $\mathbf{f}$) and for all i we have $w_i^1 \in G^p_{\ge\eta+1}$.

We define $\mathcal{F}_1^n$ inductively. Let $\mathcal{F}_1^n(\nu) = \sum_{i\in I_n} w_i^n + \sum_{b\in B_n} g_b^n$, with all $w_i^n \in G^p_{\ge\eta+n}$ and $g_b^n \in G$. We define $\mathcal{F}_1^{n+1}(\nu) = \sum_{i\in I_n} \mathcal{F}_1^1(w_i^n) + \sum_{b\in B_n} g_b^n$. So, as $G^p_p \subseteq \mathcal{A}G$, if we define $\mathcal{F}_1(\nu) := \mathcal{F}_1^{p-\eta}(\nu)$ we can conclude the proof. $\square$

4.4.2. *The expression $\mathcal{F}_2(\nu)$*

Let $\nu = ({}^{i_k}d_k, \ldots, {}^{i_1}d_1) \in G$ be an expression of $\overline{\nu} \in \mathrm{Hom}(\theta_{s_1}\cdots\theta_{s_n}, R)$. In this section we will define $\mathcal{F}_2(\nu)$, the element that we are searching for in Proposition 4.3. If $\nu\{\min m\text{-bad}\} = 0$ we define $\mathcal{F}_2(\nu) = \nu$. Now let us suppose $p := \nu\{\min m\text{-bad}\} \neq 0$. Let $\mathrm{Codom}(\nu_{p-1}) = (u_1, \ldots, u_c)$. By definition, the pth term of ν is ${}^{b-1}m$ (for some $b \in \mathbb{N}$) and there is some $a < b$ with $u_a = u_b$ and $m(u_b, u_i) = 2$ for all $a < i < b$.

By relation **a** applied several times we obtain

$$^{b-1}m = {}^{a-1}f \circ {}^{a}f \circ \cdots \circ {}^{b-3}f \circ {}^{b-1}m \circ {}^{b-3}f \circ \cdots \circ {}^{a}f \circ {}^{a-1}f. \quad (1)$$

We replace the pth term of ν by the right-hand side of the Eq. (1) and we obtain a new expression of $\overline{\nu}$, and in this new expression we apply relation 7 in the $(p+b-a-1)$th term. We obtain $\overline{\nu} = \overline{\nu_1 + \nu_2 + \nu_3}$, where $\nu_1 \in G$ and $\nu_2, \nu_3 \in G_\eta^p$ for some η. We define $\mathcal{F}_2^1(\nu) = \nu_1 + \mathcal{F}_1(\nu_2) + \mathcal{F}_1(\nu_3)$. If we write $\mathcal{F}_2^1(\nu) = \sum_i \lambda_i^1 g_i^1$, with λ_i^1 polynomials and g_i^1 elements of G, it is easy to see that $g_i^1\{\#m, j\} = \nu\{\#m, j\}$ for all i.

The following equation is trivial from the definitions:

$$\nu\{\#m, j\} = \nu\{\#m, j \text{ after min. } m\text{-bad}\} + \text{card}(\{1 \le a \le \nu\{\text{min } m\text{-bad}\} \mid d_a = m \text{ or } d_a = j\})$$

so in the lexicographical order we have the inequality

$$g_i^1\{\text{function of } m\text{-bad}\} < \nu\{\text{function of } m\text{-bad}\} \quad \text{for all } i. \quad (2)$$

If $\mathcal{F}_2^N(\nu) = \sum_i \lambda_i^N g_i^N$, with λ_i^N polynomials and g_i^N elements of G, we define $\mathcal{F}_2^{N+1}(\nu) = \sum_i \lambda_i^N \mathcal{F}_2^1(g_i^N)$, and by Eq. (2) and the fact that

$$\nu\{\text{function of } m\text{-bad}\} \ge (0, 0)$$

we conclude that there exist $N \gg 0$ such that $\mathcal{F}_2^N(\nu) = \mathcal{F}_2^{N+1}(\nu)$. We then define $\mathcal{F}_2^N(\nu) = \mathcal{F}_2(\nu)$. This means in particular that if $\mathcal{F}_2(\nu) = \sum_{\delta\in\Delta} \lambda_\delta \nu_\delta$, then $\nu_\delta\{\#m\text{-bad}\} = 0$, which proves Proposition 4.3. □

4.5. Good g-expressions

In this section we prove that if we start with an expression in G, we can restrict our attention to the case where our element ν is in G, $\nu\{\#m\text{-bad}\} = 0$ and ν is a good g-expression:

Proposition 4.6. *For every $\nu \in G$ there exist a set Δ, polynomials λ_δ and elements $\nu_\delta \in G$ such that $\overline{\nu} = \overline{\sum_{\delta\in\Delta} \lambda_\delta \nu_\delta}$ and such that*

- *$\nu_\delta\{\#m\text{-bad}\} = 0$ for all $\delta \in \Delta$.*
- *If the kth term of ν_δ is ${}^{i}f$ then the $(k+1)$th term is ${}^{i+1}f$ or ${}^{i+1}j$.*

Remark 4.7. The second condition means that we can see ν_δ as a good g-expression. In fact we can see it in many ways as a good g-expression but we will fix this ambiguity by saying that we consider it as a good g-expression of minimal length (see Definition 2.2). Another way of saying this is that when we have a substring of ν_δ that is of the form $({}^{i}m, {}^{i}j)$, then this substring belongs entirely to one $g_i(t_i, t_i')$ as defined in Definition 2.2.

Proof. We start by introducing some new relations:

- The relation

  $$(id \otimes j_s) \circ (f_{sr} \otimes id) \circ (id \otimes f_{sr}) = f_{sr} \circ (j_s \otimes id) : \theta_s\theta_s\theta_r \to \theta_r\theta_s$$

  can be deduced from **a** and **c** and will be called **c**′.
- The relation

  $$(id \otimes m_r) \circ f_{rs} = m_r \otimes id : \theta_r\theta_s \to \theta_s$$

  can be deduced from **a** and **b** and will be called **b**′.
- A commutation relation (see Section 4.2) of type $(j, f) = (f, j)$ or $(m, f) = (f, m)$ will be called a relation **x**.
- A commutation relation of type $({}^{i}f, {}^{k}f) = ({}^{k}f, {}^{i}f)$ will be called a relation **y**.

For $\nu \in G$ we define Y_ν as the set of $\nu' \in G$ such that there exist a natural number n and a sequence of expressions $(\nu_n, \ldots, \nu_2, \nu_1)$ satisfying that $\nu_1 = \nu$, $\nu_n = \nu'$ and ν_{i+1} is obtained by applying a relation **y** to ν_i for all $1 \le i \le n-1$.

We say that ν satisfies property (Q) if every $\nu' \in Y_\nu$ satisfies that none of the relations in the set $\mathbf{Q} := \{\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{x}\}$ can be applied to ν'. In other words, if $\nu' \in Y_\nu$, there is no substring of ν' that is a left X-relation (see Section 4.2) for X in the set **Q**. This property is central for the following step of the proof:

Lemma 4.8. *Let $\nu \in G$. There exists an expression $\mathcal{F}_3(\nu) \in G$, with $\overline{\mathcal{F}_3(\nu)} = \overline{\nu}$, satisfying property (Q).*

Proof. We start with a definition.

Definition 4.9. Let $\nu \in G$. If with relations of type **y** applied to ν it is possible to apply one of the relations in **Q**, we say that $\nu \in L$.

If with **y** relations applied to ν it is possible to apply one of the relations in **Q**, we do it and we call the resulting expression $\mathcal{F}_3^1(\nu)$. This is not well defined because there might be many ways of doing this, but we choose one of these ways arbitrarily. We define recursively $\mathcal{F}_3^n(\nu) = \mathcal{F}_3^1(\mathcal{F}_3^{n-1}(\nu))$.

In the following table the symbol — means that the corresponding relation decreases the corresponding $\nu\{-\}$ in the lexicographical order, the symbol -0 means that sometimes it decreases it and sometimes it maintains it equal, and the symbol 0 means that it always maintains it equal. By definition $\nu\{\text{positions of } j\text{'s}\} \in \mathbb{N}^k$, $\nu\{\#f\}$, $\nu\{f \text{ to the right}\}$, $\nu\{\text{depth } m \text{ and } j\} \in \mathbb{N}$.

	$\nu\{\text{positions of } j\text{'s}\}$	$\nu\{\#f\}$	$\nu\{f \text{ to the right}\}$	$\nu\{\text{depth } m \text{ and } j\}$
$\mathbf{c}'$	–			
$\mathbf{c}$	–			
$\mathbf{a}$	-0	–		
$\mathbf{b}$	-0	–		
$\mathbf{b}'$	-0	–		
$\mathbf{d}$	0	0	–	
$\mathbf{x}$	-0	0	-0	–
$\mathbf{y}$	0	0	0	0

As $\nu\{\text{positions of } j\text{'s}\} \geq (0, \dots, 0)$, $\nu\{\#f\} \geq 0$, $\nu\{f \text{ to the right}\} \geq 0$ and $\nu\{\text{depth } m \text{ and } j\} \geq 0$, this table shows that there exists some $N \in \mathbb{N}$ such that $\mathcal{F}_3^N(\nu) \notin L$. We put $\mathcal{F}_3(\nu) = \mathcal{F}_3^N(\nu)$, and this proves Lemma 4.8. □

Now we are able to prove the crucial lemma of this section:

Lemma 4.10. *Let $\nu \in G$. If the kth term of $\mathcal{F}_3(\nu)$ is ${}^i f$ then the $(k+1)$th term is ${}^{i+1}f$ or ${}^{i+1}j$.*

Proof. Let $\mathcal{F}_3(\nu) = ({}^{\omega_k}d_k, \dots, {}^{\omega_1} d_1)$. We use the following standard notation: if $a < b$ are two natural numbers, then $[\![a, b]\!] = \{a, a+1, a+2, \dots, b\}$.

Let $a(1) \leq b(1) < a(2) \leq b(2) < \cdots < a(r) \leq b(r)$ be such that

$$\mathcal{F}_3(\nu)[f] = \bigcup_{i=1}^{r} [\![a(i), b(i)]\!].$$

Let us fix $1 \leq l \leq r$. We only need to prove the following facts:

A. $d_{b(l)+1} = j$ and $\omega_{b(l)+1} = \omega_{b(l)} + 1$.
B. For n satisfying the inequality $a(l) \leq b(l) - n \leq b(l)$, we have $\omega_{b(l)-n} = \omega_{b(l)} - n$.

We start by proving **A**. By the definition of $b(l)$, the $(b(l)+1)$th term of $\mathcal{F}_3(\nu)$ is j or m (it cannot be an α because $\mathcal{F}_3(\nu) \in G$). As it is impossible to apply the relation $\mathbf{x}$ in $\mathcal{F}_3(\nu)$, the $b(l)$th term does not commute with the $(b(l)+1)$th term, so we have four possibilities for the $(b(l)+1)$th term:

(1) ${}^{\omega_{b(l)}}m$. (2) ${}^{\omega_{b(l)}+1}m$. (3) ${}^{\omega_{b(l)}-1}j$. (4) ${}^{\omega_{b(l)}+1}j$.

The cases (1), (2) and (3) are impossible respectively by relations $\mathbf{b}$, $\mathbf{b}'$ and $\mathbf{c}$, so we have proved **A**.

We will now prove **B** by induction on n. We suppose that we have proved it for n; we will prove it for $n+1$ if $a(l) \leq b(l) - (n+1)$. Let

$$\mathrm{Codom}(\mathcal{F}_3(\nu)_{b(l)-n-2}) = (u_1, \dots, u_q).$$

By definition of $a(l)$ we have that the $(b(l)-n-1)$th term of $\mathcal{F}_3(\nu)$ is ${}^i f$ for $i = \omega_{b(l)-n-1}$. We will prove that if $i \neq \omega_{b(l)} - n - 1$, then by applying a sequence of relations of type $\mathbf{y}$ to $\mathcal{F}_3(\nu)$ we will arrive at a new expression of $\overline{\nu}$ to which we can apply some of the relations in $\mathbf{Q}$, and this is in contradiction with Lemma 4.8.

Let us be more precise. If $i \leq \omega_{b(l)} - n - 2$ or $i \geq \omega_{b(l)} + 3$ then there is an element in $Y_{\mathcal{F}_3(\nu)}$ (see Section 4.5) to which we can apply relation $\mathbf{x}$. In fact to see this it is enough to apply the commutation relation $\mathbf{y}$ to the terms $(b(l)-n-1, b(l)-n)$, then to the terms $(b(l)-n, b(l)-n+1)$, then to the terms $(b(l)-n+1, b(l)-n+2)$, and continue like this until we apply relation $\mathbf{y}$ to the terms $(b(l)-1, b(l))$. Now we can apply relation $\mathbf{x}$ to the terms $(b(l), b(l)+1)$.

In a similar way we obtain:

- If $i = \omega_{b(l)} - n$ there is an element in $Y_{\mathcal{F}_3(\nu)}$ to which we can apply relation $\mathbf{a}$.
- If $\omega_{b(l)} - n + 1 \leq i \leq \omega_{b(l)}$ there is an element in $Y_{\mathcal{F}_3(\nu)}$ to which we can apply relation $\mathbf{e}$.
- If $i = \omega_{b(l)} + 1$ there is an element in $Y_{\mathcal{F}_3(\nu)}$ to which we can apply relation $\mathbf{c}'$.
- If $i = \omega_{b(l)} + 2$ there is an element in $Y_{\mathcal{F}_3(\nu)}$ to which we can apply relation $\mathbf{c}$.

This allows us to prove **B** and Lemma 4.10. □

Lemma 4.10 joint with the properties of $\mathcal{F}_2$ allows us to finish the proof of Proposition 4.6. □

4.6. *Elimination of the α's*

In this section we prove that we can "take out" an α_s that is "inside" an element of G in a similar way as in Section 4.4.1. We give a rigorous statement of this, in Proposition 4.12. We start by introducing some relations that we will need later:

Proposition 4.11. *We have the following equalities:*

N1 $(\mathrm{id} \otimes f_{rs}) \circ (f_{rs} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha_s) = (\alpha_s \otimes \mathrm{id}) : \theta_r \to \theta_s\theta_s\theta_r$.

N2 $(\mathrm{id}\otimes j_s)\circ(f_{sr}\otimes \mathrm{id}) = f_{sr}\circ(j_s\otimes \mathrm{id})\circ(\mathrm{id}\otimes f_{rs}) : \theta_s\theta_r\theta_s \to \theta_r\theta_s$.
N3 $(\mathrm{id}\otimes f_{rs})\circ(f_{rs}\otimes \mathrm{id})\circ(\mathrm{id}\otimes p_s) = (p_s\otimes \mathrm{id})\circ f_{rs} : \theta_r\theta_s \to \theta_s\theta_s\theta_r$.
N4 $f_{rs}\circ(\mathrm{id}\otimes \epsilon_s) = \epsilon_s\otimes \mathrm{id} : \theta_r \to \theta_s\theta_r$.
N5 $(m_r\otimes \mathrm{id})\circ p_r = (\mathrm{id}\otimes m_r)\circ p_r = \mathrm{id} : \theta_r \to \theta_r$.
N6 $j_r\circ(\mathrm{id}\otimes \epsilon_r) = j_r\circ(\epsilon_r\otimes \mathrm{id}) = \mathrm{id} : \theta_r \to \theta_r$.
N7 $(\mathrm{id}\otimes j_r)\circ(p_r\otimes \mathrm{id}) = (j_r\otimes \mathrm{id})\circ(\mathrm{id}\otimes p_r) = p_r\circ j_r : \theta_r\theta_r \to \theta_r\theta_r$.

Proof. We will give in order the relations needed to prove each one of these equations (CR means commutation relations):

- **N1**: **e**, **a**.
- **N2**: **c′**, **a**.
- **N3**: **e**, **N2**, CR, **a**.
- **N4**: CR, **e**, **a′**.
- **N5**: 2, 3.
- **N6**: CR, 2, 3.
- **N7**: 2, CR, 2, 5, CR. □

Proposition 4.12. *Let $\tau = \overline{g}\circ\xi$ with $g \in G$, $\xi =^i \alpha_s$. There exist a set Π and for each $\pi \in \Pi$ a polynomial λ_π and $g_\pi \in G$ such that*

$$\tau = \overline{\sum_{\pi\in\Pi} \lambda_\pi g_\pi}.$$

Proof. We start with a lemma.

Lemma 4.13. *To prove Proposition 4.12 it suffices to prove it for $\xi =^i p_s$ and $\xi =^i \epsilon_s$.*

Proof. We have that $\mu_1 = (\mathcal{F}_3(g), {}^i\alpha_s)$ is an expression of the morphism τ. With commutation relations we change μ_1 in μ_2, an expression where the α is as far to the left as possible. This means that μ_2 can be written in the form $({}^{i_p}d_p, \ldots, {}^{i_{\eta+1}}d_{\eta+1}, {}^i\alpha_s, {}^{i_{\eta-1}}d_{\eta-1}, \ldots, {}^{i_1}d_1)$, with $d_a \neq \alpha$ for all a and η maximal with this property. We have seven possibilities for the ${}^{i_{\eta+1}}d_{\eta+1}$: (1) ${}^{i-1}j$, (2) ${}^ij$, (3) ${}^{i+1}j$, (4) ${}^im$, (5) ${}^{i+1}m$, (6) ${}^{i+1}f$ and (7) ${}^{i-1}f$.

- In case (1), by relation 2 we reduce to the case $\xi =^i p_s$.
- In case (2), relation 6 tells us that $\tau = 0$.
- In case (3), by the definition of p_s we are in the case $\xi =^i p_s$.
- In case (4), by relation 1 we reduce to the case $\xi =^i \epsilon_s$.
- In case (5), by the definition of ϵ_s we are in the case $\xi =^i \epsilon_s$.
- In case (6), by Remark 4.7 we have that there exists c such that the $(\eta + r)$th term of μ_2 is ${}^{i+r}f$ for every $0 \le r < c$ and ${}^{i+c}j$ for $r = c$. But with the relation **N2** we can easily verify the following equality by induction:

$$ {}^{i+c}j \circ {}^{i+c-1}f \circ \cdots \circ {}^{i+1}f = ({}^{i+c-1}f \circ {}^{i+c-2}f \circ \cdots \circ {}^{i+1}f) \circ ({}^{i+1}j) \circ ({}^{i+2}f \circ {}^{i+2}f \circ \cdots \circ {}^{i+c}f). $$

  So if we replace the left-hand side of this equation by the right-hand side in the corresponding substring of μ_2, and we take the α as far to the left as possible, and we find a new expression of τ: $(g_1, {}^{i+1}j, {}^i\alpha, g_2)$, with $g_1, g_2 \in G$ and by the definition of p_s we reduce to the case $\xi =^i p_s$.
- In case (7), by the Remark 4.7 we have that the $(\eta+2)$th term is ${}^if$. So if we apply relation **N1** we arrive at a new expression μ_3 of τ: $\mu_3 = (g_1, {}^{i-1}\alpha_s, g_2)$, $g_1, g_2 \in G$, where g_1 is a good g-expression, the α is still in the ηth term and μ_3 has strictly fewer terms than μ_2. Now we repeat the process of taking in μ_3 the α as far to the left as possible and if we arrive another time at case (7), we find a corresponding μ_4. If we repeat this process enough times, finally we will arrive at one of the other six cases. This finishes the proof of the lemma. □

Proof of Proposition 4.12 for $\xi =^i p_s$. The proof of this case is very similar to the proof of Lemma 4.13; the only difference is that we use different relations. We have that $\mu_1 = (\mathcal{F}_3(g), {}^ip_s)$ is an expression of the morphism τ. With commutation relations we change μ_1 in μ_2, an expression where the p_s is as far to the left as possible. Let us say that in μ_2 we have that the ηth term is ${}^ip_s$. We have seven possibilities for the $(\eta + 1)$th term: (1) ${}^{i-1}j$, (2) ${}^ij$, (3) ${}^{i+1}j$, (4) ${}^im$, (5) ${}^{i+1}m$, (6) ${}^{i+1}f$ and (7) ${}^{i-1}f$.

- In cases (1) and (3) we apply relation **N7**, and we arrive at an expression in which the p_s is more to the left than before.
- In case (2) we have that $\tau = 0$ because the relations 5, 6 tell us that $j_s \circ p_s = 0$.
- In cases (4) and (5) we apply relation **N5** and we find an expression of τ that is in G.
- In case (6), by a similar argument to that of case (6) of Lemma 4.13, we go back to case (3).
- In case (7), by a similar argument to that of case (7) of Lemma 4.13, but using the relation **N3** instead of relation **N1**, we see that – as in cases (1), (3) and (6) – the p_s is more to the left than before. So if we repeat this process enough times, we will go back to one of the cases that are left, that is, case (2), (4) or (5).

Proof of Proposition 4.12 for $\xi =^i \epsilon_s$. We have that $\mu_1 = \mathcal{F}_3(g) \circ^i \epsilon_s$ is an expression of morphism τ. With commutation relations we change μ_1 to μ_2, an expression where the ϵ_s is as far to the left as possible. Let us say that in μ_2 we have that the ηth term is ${}^i\epsilon_s$. We have five possibilities for the $(\eta + 1)$th term: (1) ${}^{i-1}j$, (2) ${}^ij$, (3) ${}^im$, (4) ${}^if$ and (5) ${}^{i-1}f$.

- In cases (1) and (2), using relation **N6** we find an expression of τ that is in G.
- In case (4), by a similar argument to that of case (6) in Lemma 4.13, we go back to case (2).
- In case (5), by a similar argument to that of case (7) in Lemma 4.13, but using relation **N4** instead of relation **N1** we see that the ϵ_s is more to the left than before. So, if we repeat enough times we will go back to one of the cases that are left.
- Case (3) is treated in Section 4.4.1. □

By using Proposition 4.12 repeatedly we have the following:

Corollary 4.14. *Let ν be an R-expression. There exist a set Π and for each $\pi \in \Pi$ a polynomial λ_π and an element $g_\pi \in G$ such that*

$$\overline{\nu} = \overline{\sum_{\pi \in \Pi} \lambda_\pi g_\pi}.$$

4.7. Good order

The purpose of this section is to change an R-expression that is a good g-expression into an R-expression that is a good g-expression in the good order.

Proposition 4.15. *Let $a > b$ and $\alpha, \beta \in \{m, ch, cch\}$. For all $(a', b') \in \mathbb{N}^2$ such that $\alpha(a, a')$ and $\beta(b, b')$ are defined, there exist integers c, c', d and d' with $c < d$ such that $\alpha(a, a') \circ \beta(b, b') = \beta(c, c') \circ \alpha(d, d')$.*

Proof.
- Let us consider first the case $\beta = \text{ch}$ and $\alpha = m$ (the case $\beta = \text{cch}$ and $\alpha = m$ is similar). We have two possible cases.
  - If $a \geq b + b' + 2$ then $m(a, 0)$ commutes with $\text{ch}(b, b')$, so with commutation relations we obtain

    $$m(a, 0) \circ \text{ch}(b, b') = \text{ch}(b, b') \circ m(a + 1, 0).$$
  - If $b + b' + 1 \geq a > b$ then we have

    $$\begin{aligned} m(a, 0) \circ \text{ch}(b, b') &= \{m^a\} \circ j^b \circ f^{b+1} \circ f^{b+2} \circ \cdots \circ f^{b+b'} \\ &= j^b \circ f^{b+1} \circ \cdots \circ f^{a-2} \circ \{(m^{a+1}) \circ f^{a-1}\} \circ \cdots \circ f^{b+b'} \\ &= j^b \circ f^{b+1} \circ \cdots \circ f^{a-2} \circ \{m^a\} \circ f^{a-1} \circ \cdots \circ f^{b+b'-1} \\ &= j^b \circ f^{b+1} \circ \cdots \circ f^{b+b'-1} \circ \{m^a\} \\ &= \text{ch}(b, b' - 1) \circ m(a, 0). \end{aligned}$$

  In each equation we have marked inside $\{-\}$ the substring that we changed. The second and fourth equalities are a consequence of commutation relations and in the third equality we applied relation **b** inside the parentheses.
- The case $\alpha = m$, ch or cch and $\beta = m$ is easy because α commutes with β.
- The last case is $\alpha = \text{ch}$ or cch and $\beta = \text{ch}$ or cch. We will only treat the case $\alpha = \text{ch}$ and $\beta = \text{ch}$; the other ones are similar. We will prove that $\text{ch}(a, a') \circ \text{ch}(b, b') = \text{ch}(b, b') \circ \text{ch}(a, a' + 1)$. For this we need a preliminary lemma.

Lemma 4.16. *If $f =^{i_q} f \circ \cdots \circ^{i_1} f$, $g =^{k_p} f \circ \cdots \circ^{i_1} f \in \text{Hom}(\theta_{s_1} \cdots \theta_{s_n}, \theta_{u_1} \cdots \theta_{u_n})$, then $f = g$.*

Proof. Relations **a** and **e** are exactly the relations defining the symmetric group. □

Now we can prove $\text{ch}(a, a') \circ \text{ch}(b, b') = \text{ch}(b, b') \circ \text{ch}(a, a' + 1)$, where we suppose of course that $\text{ch}(a, a')$ do not commute with $\text{ch}(b, b')$, i.e.

$$a \leq b + b' + 1.$$

$$\begin{aligned} \text{ch}(a, a') \circ \text{ch}(b, b') &= (j^a, f^{a+1}, \ldots, f^{a+a'}, \{j^b\}, f^{b+1}, \ldots, f^{b+b'}) \\ &= (j^a, j^b, \{f^{a+2}, \ldots, f^{a+a'+1}, f^{b+1}, \ldots, f^{b+b'}\}) \\ &= (\{j^a\}, j^b, f^{b+1}, \ldots, f^{b+b'+1}, f^{a+1}, \ldots, f^{a+a'+1}) \\ &= (j^b, f^{b+1}, \ldots f^{a-2}, \{j^a, f^{a-1}\}, f^a, \ldots, f^{b+b'+1}, f^{a+1}, \ldots, f^{a+a'+1}) \\ &= (j^b, f^{b+1}, \ldots f^{a-2}, f^a, j^{a-1}, \{f^a, f^a\}, \ldots, f^{b+b'+1}, f^{a+1}, \ldots, f^{a+a'+1}) \\ &= (j^b, f^{b+1}, \ldots f^{a-2}, f^{a-1}, \{j^{a-1}\}, \ldots, f^{b+b'+1}, f^{a+1}, \ldots, f^{a+a'+1}) \\ &= \text{ch}(b, b') \circ \text{ch}(a, a' + 1). \end{aligned}$$

In each equation we have marked inside $\{-\}$ the substring that we changed. The second, fourth and seventh equalities are due to a sequence of commutation relations, the third equality is a consequence of Lemma 4.16, in the fifth equality we applied relation **c** and in the sixth equality we applied relation **a**. □

By Proposition 4.15 applied repeatedly, if ν is a good g-expression we can find another expression, that we will name $\mathcal{F}_4(\nu)$, satisfying that is a good g-expression in the good order and such that $\overline{\mathcal{F}_4(\nu)} = \overline{\nu}$.

4.8. Definition of $\mathcal{F}_5(\nu)$

In this section we define the element $\mathcal{F}_5(\nu)$. We will see in the next section that this is the element that we are looking for to finish the proof of Theorem 3.3. If $\tau = \sum_i \lambda_i g_i$, with I a finite set, λ_i polynomials and $g_i \in G$, we define

$$\mathcal{F}_k(\tau) = \sum_i \lambda_i \mathcal{F}_k(g_i),$$

for $2 \leq k \leq 4$. Let $\nu \in G$. We define $\mathcal{F}_5^1(\nu) = \mathcal{F}_4\mathcal{F}_3\mathcal{F}_2(\nu)$. This is well defined, because $\mathcal{F}_2(\nu)$ is a linear combination of elements of G and $\mathcal{F}_3$ applied to an element of G gives a good g-expression.

We define inductively $\mathcal{F}_5^n(\nu)$: if $\mathcal{F}_5^n(\nu)\{\#m\text{-bad}\} \neq 0$ we define $\mathcal{F}_5^{n+1}(\nu) = \mathcal{F}_5^1\mathcal{F}_5^n(\nu)$ and if $\mathcal{F}_5^n(\nu)\{\#m\text{-bad}\} = 0$, we define $\mathcal{F}_5^{n+1}(\nu) = \mathcal{F}_5^n(\nu)$.

Proposition 4.17. *$\mathcal{F}_5^n(\nu)$ stabilizes for n large. We call this common value $\mathcal{F}_5(\nu)$.*

Proof. Let us suppose that $\mathcal{F}_5^n(\nu)$ does not stabilize for n large. This means that for all $n \in \mathbb{N}$, we have $\mathcal{F}_5^n(\nu)\{\#m\text{-bad}\} \neq 0$. So we apply relation 7 infinitely many times in this process. When we apply relation 7 to an expression we obtain a sum of three generalized expressions, $\nu = \nu_1 + \nu_2 + \nu_3$, with $\nu_1 \in G$ and $\nu_2, \nu_3 \in G_\eta^p$ for some integers p and η. The definition of $\nu\{m, j$ equal to left$\}$ can be trivially generalized to $\nu \in G_\eta^p$ and so it is easy to see that

$$\nu\{m, j \text{ equal to left}\} > \nu_i\{m, j \text{ equal to left}\} \quad \text{for all } 1 \leq i \leq 3.$$

The other relations used in $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ do not change $\nu\{m, j$ equal to left$\}$. To see this, we make a brief review of all relations used in defining $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$:

- $\mathcal{F}_2$: commutation relations, **a** in the opposite sense, 7, **g**, **f**.
- $\mathcal{F}_3$: **a**, **b**, **b**′, **c**, **c**′, **d**, **x**, **y**.
- $\mathcal{F}_4$: commutation relations, **a**,**b**, **c**, **d**.

So, as $\nu\{m, j$ equal to left$\} \geq 0$, we have a contradiction, and this allows us to conclude the proof. □

Remark 4.18. As $\mathcal{F}_4(\nu)$ is a good g-expression in the good order, $\mathcal{F}_5(\nu)$ is a good g-expression in the good order satisfying $\mathcal{F}_5(\nu)\{\#m\text{-bad}\} = 0$.

4.9. End of the proof of Theorem 3.3

We start with a useful reformulation of Proposition 2.4 in the new terminology that we have introduced:

Corollary 4.19. *Let $(W, \mathcal{S})$ be a right-angled Coxeter system. Let $(s_1, \ldots, s_n) \in \mathcal{S}^n$. We have the equality of sets:*
$FL(s_1, \ldots, s_n) = \{\overline{\nu}$ such that ν is a good g-expression in $\mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, R)$ in the good order, with $\nu\{\#m\text{-bad}\} = \nu\{\#j\text{-bad}\} = 0\}$.

Using Lemma 4.2 we proved that in order to prove Theorem 3.3 we only need to prove the following proposition:

Proposition 4.20. *Let $(s_1, \ldots, s_n) \in \mathcal{S}^n$. If $f \in \mathrm{Hom}(\theta_{s_1} \cdots \theta_{s_n}, R)$ then there exists a finite set I and for $i \in I$, $a_i \in FL(s_1, \ldots, s_n)$ and λ_i, polynomials such that $f = \sum_i a_i \lambda_i$.*

As f is a linear combination of R-expressions over the field k, we can restrict to the case where f is an R-expression. By Corollary 4.14 we can restrict to the case $f \in G$. By Proposition 4.17 and Remark 4.18 we can restrict to the case where f is a good g-expression in the good order and with $\nu\{\#m\text{-bad}\} = 0$. So, the following lemma allows us to conclude the proof of Theorem 3.3:

Lemma 4.21. *If ν is an R-expression that is a good g-expression in the good order and $\nu\{\#m\text{-bad}\} = 0$, then $\nu\{\#j\text{-bad}\} = 0$.*

Proof. Suppose $\nu\{\#j\text{-bad}\} \neq 0$. Let us put $\eta = \nu\{\max. j\text{-bad}\}$. Let ${}^z j_s$ be the ηth term of ν. The $(\eta + 1)$th term of ν cannot be ${}^z m_s$, because $\nu\{\#m\text{-bad}\} = 0$.

Let us define by induction the natural numbers N_p for $0 \leq p \leq r(\nu)$, where $r(\nu)$ is a (maybe infinite) number that will be defined in the process. We will construct N_p in such a way that the N_pth element of $\mathrm{Codom}(\nu_{\eta+p})$ will always be the same element $s \in \mathcal{S}$. We define $N_0 = z + 1$. Let us suppose that we have defined N_{p-1}.

- Suppose that the $(\eta + p)$th term of ν is ${}^i m$. Then we define $N_p = N_{p-1}$.
- Suppose that the $(\eta + p)$th term of ν is ${}^i j$. If $i > N_{p-1} - 1$ then $N_p = N_{p-1}$ and if $i = N_{p-1} - 1$ then $p - 1 := r(\nu)$.
- Suppose that the $(\eta + p)$th term of ν is ${}^i f$. If $i \notin \{N_{p-1} - 1, N_{p-1} - 2\}$ then $N_p = N_{p-1}$. If $i = N_{p-1} - 1$ then $N_p = N_{p-1} + 1$ and if $i = N_{p-1} - 2$ then $N_p = N_{p-1} - 1$.

If for all p such that the $(\eta+p)$th term of ν is ${}^i j$ we have that $i \neq N_{p-1} - 1$, we define $r(\nu) = \infty$. The integers N_p are well defined because ν is a good g-expression in the good order. By construction, the N_pth element of $\mathrm{Codom}(\nu_{\eta+p})$ is always the same element $s \in \mathcal{S}$. The fact that the N_pth element of $\mathrm{Codom}(\nu_{\eta+p})$ is of left type is a consequence of the same assertion for $p = 0$.

As ν is an R-expression and the $(\eta+p)$th term of ν cannot be ${}^{N_p-1}m$ (because $\nu\{\#m\text{-bad}\}=0$), we have that there is some p such that the $(\eta+p)$th term of ν is ${}^{i}j$ with $i=N_{p-1}-1$ (i.e. $r(\nu)$ is a finite number). This contradicts the definition of $\nu\{\text{max. } j\text{-bad}\}$, so we conclude that $\nu\{\#j\text{-bad}\}=0$, and this proves Lemma 4.21, Proposition 4.20 and Theorem 3.3. □

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