*Proc. London Math. Soc.* (3) 109 (2014) 1264–1280 
doi:10.1112/plms/pdu022

# Standard objects in 2-braid groups

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Abstract

For any Coxeter system, we establish the existence (conjectured by Rouquier) of analogues of standard and costandard objects in 2-braid groups. This generalizes a known extension vanishing formula in the BGG category $\mathcal{O}$.

## 1. *Introduction*

In [**17**], Rouquier introduces a categorification of (a quotient of) the Artin braid group associated to a Coxeter system. He calls the resulting monoidal category the 2-braid group. It occurs in the study of categories of representations of semi-simple Lie algebras, affine Lie algebras, reductive algebraic groups, quantum groups, etc. On the other hand, it has been used to categorify the HOMFLYPT polynomial of a link (see [**9**]).

Let us be more precise. Let $(W, \mathcal{S})$ be a Coxeter system and $V$ be the geometric representation of $W$ over the complex numbers. Let $R$ be the regular functions on $V$, graded such that $\deg V^* = 2$. The group $W$ acts on $V$, so by functoriality it acts on $R$. For $s \in \mathcal{S}$, let $R^s$ be the subspace of $R$ of $s$-fixed points. Consider the following complexes of graded $R$-bimodules:

$$F_s = \cdots \longrightarrow 0 \longrightarrow R \otimes_{R^s} R(1) \longrightarrow R(1) \longrightarrow 0 \longrightarrow \cdots,$$
$$F_{s^{-1}} = \cdots \longrightarrow 0 \longrightarrow R(-1) \longrightarrow R \otimes_{R^s} R(1) \longrightarrow 0 \longrightarrow \cdots.$$

Here (1) denotes the grading shift functor (normalized so that $R(1)$ is generated in degree $-1$). In both cases, $R \otimes_{R^s} R(1)$ is the degree zero term of the complex and the non-trivial differentials of $F_s$ and $F_{s^{-1}}$ are the unique non-zero maps of degree zero (which are well defined up to a scalar).

Let $K^b(R\text{-Mod-}R)$ denote the homotopy category of bounded complexes of graded $R$-bimodules, which is a monoidal category under tensor product of complexes. Let $B_W$ be the Artin braid group associated to the Coxeter system $(W, \mathcal{S})$. Given any word $\sigma$ in $\mathcal{S}$ and $\mathcal{S}^{-1}$ one can consider the corresponding product of the complexes $F_s$ and $F_{s^{-1}}$ above. In [**17**], Rouquier shows that the corresponding complex $F_\sigma$ in the homotopy category only depends up to isomorphism on the image of $\sigma$ in the Braid group $B_W$. One obtains in this way a 'weak categorification': a homomorphism from $B_W$ to the set of isomorphisms classes of complexes in $K$ (which has the structure of a monoid induced from the monoidal structure on $K$).

Moreover, Rouquier [**17**] shows that any two expressions for $\sigma \in B_W$ give rise to canonically isomorphic complexes. Hence, one obtains a 'strict categorification'. That is, one has a monoidal functor

$$F : \Omega B_W \longrightarrow K^b(R\text{-Mod-}R),$$

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Received 29 June 2012; revised 10 October 2013; published online 13 June 2014.

2010 *Mathematics Subject Classification* 20F36 (primary), 20G05, 17B10 (secondary).

Nicolás Libedinsky would like to acknowledge the support of the von Humboldt Foundation and 'Proyecto Fondecyt Iniciación 11121118'.

where $\Omega B_W$ is the monoidal category associated to $B_W$: the objects of $\Omega B_W$ are the elements $\sigma \in B_W$, the morphisms are given by $\mathrm{Hom}(\sigma, \sigma') = \emptyset$ if $\sigma \neq \sigma'$ and $\mathrm{End}(\sigma) = \{\mathrm{id}\}$, and the monoidal structure is given by the group structure on $B_W$.

Let $\mathfrak{B}_W$ denote the full subcategory of $K^b(R\text{-Mod-}R)$ consisting of all objects isomorphic to objects in the image of $F$. Rouquier calls $\mathfrak{B}_W$ the 2-*braid group* of the Coxeter system $(W, \mathcal{S})$. (Strictly speaking, what Rouquier calls the 2-braid group is a strict monoidal category monoidally equivalent to $\mathfrak{B}_W$.) As we have seen, the decategorification of $\mathfrak{B}_W$ is a quotient of $B_W$. He conjectures that the decategorification is exactly $B_W$. This conjecture is true for the topological braid group (that is, when $W$ is the symmetric group) by work of Khovanov and Seidel [**10**].

As we have seen, when viewed as a monoidal category the morphism spaces in $\Omega B_W$ are boring; all information is already contained in the structure of $B_W$ as a group. This is far from true for $\mathfrak{B}_W$. There is a rich and at present poorly understood structure in the morphism spaces of $\mathfrak{B}_W$.

The reader seeking an analogy might like to think about the braid group of type $A_n$, where one can view braids topologically. In this case, one has a natural categorification, the category of 'braid cobordisms': objects are topological braids and morphisms are certain cobordisms (see [**2**] or the introduction of [**11**]). In this case, there is a monoidal functor from the category of braid cobordisms to the 2-braid group (see [**5**, **11**]). Unfortunately, certain generating cobordisms (either the birth or death of a single crossing) are necessarily mapped to zero. So the (topological) category of braid cobordisms and the (algebraic) 2-braid group seem to be quite different.

This paper can be seen as a first attempt to understand the homomorphisms in $\mathfrak{B}_W$. More precisely, we explain how the canonical section of the projection

$$B_W \twoheadrightarrow W$$

allows one to define 'standard' and 'costandard' objects in $\mathfrak{B}_W$. Given $w \in W$, let $\sigma \in B_W^+$ denote the canonical positive lift of $w$ in $B_W$ and define $F_w = F_\sigma$. Our main theorem is the following.

THEOREM 1.1. *For $x, y \in W$, we have*

$$\mathrm{Hom}(F_x, F_{y^{-1}}^{-1}[i]) \cong \begin{cases} R & \text{if } x = y \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our theorem was conjectured in Rouquier's ICM address [**18**, 4.2.1]. Hence, we refer to this equation as Rouquier's formula. This formula generalizes for all Coxeter groups the important formula in BGG category $\mathcal{O}$

$$\mathrm{Ext}^i(\Delta(w), \nabla(v)) = 0 \quad \text{if } w \neq v \in W \text{ or } i \neq 0, \tag{1}$$

where $\Delta(w)$ and $\nabla(v)$ are, respectively, the standard and costandard objects in (the principal block of) category $\mathcal{O}$.

In this paper, we establish Theorem 1.1 for all Coxeter groups, thus establishing analogues of standard and costandard objects. Let $\mathcal{B}$ denote the additive category of Soergel bimodules (see Subsection 2.2), and $K^b(\mathcal{B})$ denote its homotopy category. If $W$ is a Weyl group, then $K^b(\mathcal{B})$ is closely related to both the derived category of the principal block of category $\mathcal{O}$ (see Subsection 2.4) and to the derived category of mixed equivariant sheaves on the flag variety (see [**19**]). In the first instance, the complexes $F_x$ describe tilting resolutions of standard objects (see Lemma 2.2); in the second instance, they describe the subquotients in the weight filtrations of standard sheaves (see [**23**, Subsection 3.5]).

Remark 1.2. The conjecture in [**18**] has a misprint, it is trivially false as stated: take, for example, $b = sr$ and $b' = rs$, with $s \neq r \in \mathcal{S}$. Theorem 1.1 is the correct formulation of this conjecture.

Remark 1.3. For $i = 0$, Theorem 1.1 follows directly from the construction of the light leaves basis in [**12**].

As explained by Rouquier [**18**], proving Theorem 1.1 should shed light on the search for a presentation of the monoidal category of Soergel bimodules by generators and relations. Indeed unpacking Theorem 1.1 gives many non-trivial lifting properties of morphisms between Soergel bimodules (this is because if $\mathcal{B}$ is the category of Soergel bimodules as above, then $\mathfrak{B}_W \subseteq K^b(\mathcal{B})$). Note, however, that such a generators and relations description for the category of Soergel bimodules has recently been obtained along different lines by Elias and the second author [**6**] (following work of the first author [**14**], Elias–Khovanov [**4**] and Elias [**3**]).

On the other hand, it is desirable to have a generators and relations description for the monoidal category $\mathfrak{B}_W$. At present, this seems like a difficult problem, however, we hope that Theorem 1.1 (as well as the notion of $\Delta$ and $\nabla$-exact complexes) provide a stepping stone towards such a description.

This paper is structured as follows. Section 2 is intended as an introduction to understand where Rouquier's formula comes from and why Theorem 1.1 can be seen as a generalization of the Ext formula (1) in category $\mathcal{O}$. In Subsection 2.1, we fix some general notation. In Subsection 2.2, we give preliminaries about Soergel bimodules. In Subsection 2.3, we introduce the 2-braid group. Subsection 2.4 recalls some basic facts about BGG category $\mathcal{O}$ and in Subsection 2.5 we prove that Theorem 1.1 follows for Weyl groups from (1). This last result is a particular case of Theorem 1.1 but we hope that this section helps the reader gain some understanding of where Rouquier's formula comes from.

Section 3 is devoted to the proof of Theorem 1.1, so we work in the context of arbitrary Coxeter groups. In Subsection 3.1, we recall that Soergel bimodules are filtered by geometrically defined submodules. To this filtration, one associates 'subquotient functors' from the category $\mathcal{B}$ of Soergel bimodules to the category of graded $R$-bimodules. Soergel's Hom formula says that one can recover the Hom space between two Soergel bimodules by knowing the Hom spaces between the successive subquotients of these filtrations (see (11)).

In Subsection 3.2, we introduce the notion of $\Delta$- and $\nabla$-exact complexes. Roughly speaking, these are complexes which have good exactness properties under the 'subquotient functors' mentioned above. Using Soergel's Hom formula, we prove that the Hom space between a $\Delta$-exact complex and any complex of Soergel bimodules in the homotopy category is zero. In Subsection 3.3, we prove the main technical result used to prove Theorem 1.1, namely certain augmentations of the complexes $F_\sigma$ (respectively, $F_\sigma^{-1}$) are $\Delta$ (respectively, $\nabla$-exact) if $\sigma$ is a positive lift of an element of $W$. This is a very strong property. Indeed, it appears to be if and only if, though we cannot prove this. Finally, we use the triangulated category structure of the homotopy category in Subsection 3.5 to conclude the proof of Theorem 1.1.

## 2. *Preliminaries*

### 2.1. *Graded bimodules and their homotopy categories*

Given graded algebras $R$, $S$ over a field $k$, we denote by $R$-Mod and $R$-Mod-$S$ the categories of $\mathbb{Z}$-graded left $R$-modules and $\mathbb{Z}$-graded $(R, S)$-bimodules, respectively. (The capitalized '$M$' is intended to remind us that we are considering graded modules.) Morphisms in $R$-Mod and

$R$-Mod-$S$ are those morphisms of bimodules which preserve the grading (that is, are of degree zero). We write $\hom_{R\text{-Mod}}$, $\hom_{R\text{-Mod-}S}$ (or hom if the context is clear) for homomorphisms in these categories. Given a graded (bi)module $M = \bigoplus_{i\in\mathbb{Z}} M_i$, we define the shifted module $M(n)$ by $M(n)_i = M_{n+i}$ and set

$$\mathrm{Hom}(M,N) = \bigoplus_{i\in\mathbb{Z}} \hom(M, N(i))$$

to be the graded vector space of all (bi)module homomorphisms between $M$ and $N$. Let us emphasize that $\mathrm{Hom}(M,N)$ is only used to simplify notation at some points; it does not refer to the morphisms in any category that we consider in this paper.

Given a Laurent polynomial with positive coefficients $P = \sum a_i v^i \in \mathbb{N}[v, v^{-1}]$ and a graded (bi)module $M$, we set

$$P \cdot M := \bigoplus M(i)^{\oplus a_i}.$$

Given $M, N \in R$-Mod-$R$, we denote their tensor product simply by juxtaposition: $MN := M \otimes_R N$. This tensor product makes $R$-Mod-$R$ into a monoidal category.

Given an additive category $\mathcal{A}$, we denote by $K^b(\mathcal{A})$ its homotopy category of bounded complexes, which is obtained as the quotient of the category of bounded chain complexes by the ideal of null-homotopic morphisms. We use upper indices to indicate the terms of a complex, and all chain complexes will be cohomological. That is, an object $A$ of $K^b(\mathcal{A})$ is a complex of the form

$$\cdots \longrightarrow A^i \to A^{i+1} \longrightarrow \cdots,$$

where each $A^i \in \mathcal{A}$ and only finitely many $A^i$ are non-zero. If $\mathcal{A}$ is in addition a monoidal category, then we obtain an induced monoidal structure on $K^b(\mathcal{A})$ given by tensor product of complexes. Again, we denote the operation of tensor product by juxtaposition.

Given $A, B \in K^b(\mathcal{A})$, we write $\hom_K(A,B)$ for the homomorphisms in $K^b(\mathcal{A})$, and $\hom^\bullet(A,B)$ for the total complex of the double complex with $(i,j)$th-term $\hom(A^i, B^j)$ and differentials induced by the differentials on $A$ and $B$ (see [**8**, 11.7]). We have $\hom_K(A,B) = H^0(\hom^\bullet(A,B))$.

We will primarily be concerned with the homotopy category $K^b(R\text{-Mod-}R)$ of graded bimodules over a graded ring $R$, in which case we always assume that all differentials are of degree zero. Given $A, B \in K^b(R\text{-Mod-}R)$, we denote by $\mathrm{Hom}^\bullet(A,B)$ the total complex of the double complex with $(i,j)$th-term $\mathrm{Hom}(A^i, B^j)$ as above. As with bimodules, we write

$$\mathrm{Hom}_K(A,B) = \bigoplus_{i\in\mathbb{Z}} \hom(A, B(i)),$$

for the graded vector space of homotopy classes of morphisms of complexes of all degrees. We have $\mathrm{Hom}_K(A,B) = H^0(\mathrm{Hom}^\bullet(A,B))$.

Given $A \in K^b(\mathcal{A})$ and any $i \in \mathbb{Z}$, we have a distinguished triangle [**8**, Exercise 11.2]

$$w_{\geqslant i} A \longrightarrow A \longrightarrow w_{<i} A \xrightarrow{[1]}, \tag{2}$$

where $w_{\geqslant i}A$ (respectively, $w_{<i}A$) denote the 'stupid truncations' of $A$: $(w_{\geqslant i}A)^j = A^j$ if $j \geqslant i$ and is zero otherwise, whilst $(w_{<i}A)^j = A^j$ if $j < i$ and is zero otherwise.

## 2.2. *Soergel bimodules*

We start by recalling some aspects of Soergel bimodules, as explained in [**22**] and its relation to Rouquier's categorification of braid groups, as explained in [**17**]. Throughout, $(W, \mathcal{S})$ denotes a Coxeter system and $\mathcal{T} = \bigcup_{x\in W} x\mathcal{S}x^{-1}$ denotes the reflections in $W$. Let $\ell : W \to \mathbb{N}$ denote the length function and $\leqslant$ denote the Bruhat order on $W$.

Recall that a representation $V$ of $W$ is said to be reflection faithful if it is both faithful and has the property that an element $w \in W$ fixes a hyperplane if and only if it is a reflection. For example, the geometric representation of any finite Weyl group is reflection faithful, whereas this is never true for an affine Weyl group. It is known [**22**] that any Coxeter group has a reflection faithful representation defined over the real numbers. Throughout, we let $V$ denote a fixed reflection faithful representation over a field of characteristic $\neq 2$.

We let $R$ denote the regular functions on $V$ which we view as a graded ring with $\deg V^* = 2$. Alternatively, we can view $R$ as the symmetric algebra on $V^*$.

Throughout, we write $B_s := R \otimes_{R^s} R(1)$. The category $\mathcal{B}$ of Soergel bimodules is the smallest additive monoidal Karoubian strict subcategory of $R$-Mod-$R$ which contains $B_s$ for all $s \in \mathcal{S}$ and is stable under arbitrary shifts. The reader scared by so many adjectives will probably be happier with the following equivalent definition: $\mathcal{B}$ is the full subcategory of $R$-Mod-$R$ with objects those bimodules isomorphic to direct sums of graded shifts of direct summands of bimodules of the form $B_s B_t \cdots B_u$ for $s, t, \ldots, u \in \mathcal{S}$.

REMARK 2.1. At various points in our argument, it is more convenient to work with a reflection faithful representation when considering various filtrations on Soergel bimodules. However, using [**13**, Théorème 2.2] one can usually deduce theorems which are also valid for the geometric representation once one has proved them for a reflection faithful representation. (The key technical point is that the reflection faithful representation $V$ defined by Soergel has a subrepresentation $V_{\text{geom}} \subseteq V$ isomorphic to the geometric representation.) In particular, having established Theorem 1.1 for Soergel bimodules built using $V$ it is easy to conclude, using the results of [**13**], that it also holds if instead we had used $V_{\text{geom}}$.

## 2.3. *The 2-braid group*

We will explain in more detail the construction of Rouquier complexes. Throughout this paper, we use

$$K = K^b(R\text{-Mod-}R)$$

to denote the homotopy category of bounded complexes of graded $R$-bimodules.

For a reflection $t \in \mathcal{T}$, consider $\alpha_t \in V^*$ an equation of the hyperplane of $V$ fixed by $t$. The $\alpha_t$ are unique up to non-zero scalar, we fix them arbitrarily. In equations, $\ker(\alpha_t) = V^t$.

For $s \in \mathcal{S}$, consider the graded $R$-bimodule morphism $\eta_s : R \to B_s(1)$ defined by the equation $\eta_s(1) = \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1)$, and the multiplication morphism $m_s : B_s \to R(1)$. We define the complexes of graded $R$-bimodules:

$$F_s = \cdots \longrightarrow 0 \longrightarrow B_s \xrightarrow{m_s} R(1) \longrightarrow 0 \longrightarrow \cdots$$

and

$$F_s^{-1} = \cdots \longrightarrow 0 \longrightarrow R(-1) \xrightarrow{\eta_s} B_s \longrightarrow 0 \longrightarrow \cdots,$$

where in both complexes $B_s$ sits in complex degree zero (that is, $F_s^0 = (F_s^{-1})^0 = B_s$). It is straightforward to verify that in $K$ we have isomorphisms

$$F_s^{-1} F_s \cong F_s F_s^{-1} \cong R,$$

which justifies the notation.

Recall that the braid group $B_W$ of the Coxeter system $(W, \mathcal{S})$ is defined by the generators $\mathbf{S} = \{\mathbf{s}\}_{s \in \mathcal{S}}$ and relations

$$\underbrace{\mathbf{sts}\cdots}_{m_{st}\text{ terms}} \cong \underbrace{\mathbf{tst}\cdots}_{m_{st}\text{ terms}}.$$

Let $B_W^+$ denote the submonoid of $B_W$ generated by $\mathbf{S}$.

As explained in Section 1, in [**17**] Rouquier proves that for every two decompositions of an element of $B_W$ in a product of the generators and their inverses there exist a canonical isomorphism in $K$ between the corresponding product of $F_s$. If $\sigma$ is an element of $B_W$ as in Section 1, then we denote by $F_\sigma$ the corresponding element in $K$. Let $\sigma \in B_W^+$ be the canonical positive lift of $w \in W$. Then we define $F_w = F_\sigma$ and $E_w = F_{w^{-1}}^{-1}$.

### 2.4. *Review of category $\mathcal{O}$*

Here we give a very quick review of the facts that we will need of category $\mathcal{O}$. For more details, see [**7**].

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be, respectively, a complex semi-simple Lie algebra, a Borel and Cartan subalgebra and $W$ be the Weyl group. Let $\mathcal{O}$ be the Bernstein–Gelfand–Gelfand category of finitely generated $\mathfrak{h}$-diagonalizable and locally $\mathfrak{b}$-finite $\mathfrak{g}$-modules.

For all $\lambda \in \mathfrak{h}^*$, we have a standard module, the *Verma module* $\Delta(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ denotes the irreducible $\mathfrak{h}$-module with weight $\lambda$ inflated via the surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \cong \mathfrak{h}$ to a module over the Borel subalgebra. The module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$ and we denote $P(\lambda)$ its projective cover. Let $D$ be a duality on $\mathcal{O}$ that fixes simple modules (up to isomorphism), we put $\nabla(\lambda) = D\Delta(\lambda)$. Recall that $T$ is a *tilting object* in the category $\mathcal{O}$ if and only if $T$ and $DT$ have filtrations by $\Delta$-modules.

The dot-action of the Weyl group $W$ on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho$ is the half-sum of the positive roots. For $\lambda \in \mathfrak{h}^*$, we denote by $\mathcal{O}_\lambda$ the full subcategory of $\mathcal{O}$ with objects the modules killed by some power of the maximal ideal $\mathrm{Ann}_{\mathcal{Z}}\Delta(\lambda)$ of the centre $\mathcal{Z}$ of $\mathcal{U}(\mathfrak{g})$. This yields a decomposition

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\cdot)} \mathcal{O}_\lambda.$$

The subcategories $\mathcal{O}_\lambda$ corresponding to integral weights are indecomposable, though this is not true for general weights.

Consider $\mu, \lambda \in \mathfrak{h}^*$ such that $\lambda - \mu$ is an integral weight. Let $E(\mu - \lambda)$ be the finite-dimensional irreducible $\mathfrak{g}$-module with extremal weight $\mu - \lambda$. Then, we define the translation functor $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu, M \mapsto \mathrm{pr}_\mu(E(\mu - \lambda) \otimes M)$, where $\mathrm{pr}_\mu$ is the projection functor onto the block $\mathcal{O}_\mu$. The functors $T_\lambda^\mu$ and $T_\mu^\lambda$ are left and right adjoints to each other.

If we choose $\mu$ a singular weight with $\{1, s\}$ the stabilizer under the dot-action of the Weyl group, and $s$ a simple reflection, then we can define the wall-crossing functor as the composition $\theta_s = T_\mu^0 T_0^\mu$.

Let $W$ be a Weyl group and $R$ be as in Subsection 2.2. Let $C = R/(R_+^W)$ be the coinvariant algebra. We identify $R/R_+$ with the ground field $\mathbb{C}$. In [**20**, Endomorphismensatz], Soergel proves that there exists an isomorphism $C \cong \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot 0))$, and in [**15**, Subsection 2.3] or [**1**] it is proved that the exact functor

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}}(P(w_0 \cdot 0), -) : \mathcal{O}_0 \longrightarrow C - \mathrm{mod}$$

is fully faithful on tilting objects (this is an alternate version of [**20**, Struktursatz 9]). From now on, the objects $P(w \cdot 0), \Delta(w \cdot 0)$ and $\nabla(w \cdot 0)$ will be denoted by $P(w), \Delta(w)$ and $\nabla(w)$.

### 2.5. *Rouquier's formula for Weyl groups*

In this section, we prove Theorem 1.1 for Weyl groups using formula (1) in Section 1. Of course, it is a corollary of Theorem 1.1 proved in Section 3, but we consider this section important to understand where this formula comes from. Using the notation we have introduced, we restate Rouquier's formula. If $w, v \in W$, then we have the following isomorphisms:

$$\mathrm{Hom}_K(F_w, E_v[i]) \cong \begin{cases} R & \text{if } w = v \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the bounded complex of functors from $\mathcal{O}_0$ to itself,

$$\Phi_s = \cdots \longrightarrow 0 \longrightarrow \theta_s \longrightarrow \mathrm{Id} \longrightarrow 0 \longrightarrow \cdots,$$

where $\theta_s$ is in degree zero and the map from $\theta_s$ to the identity functor is the counit of the adjunction. Since the functors involved are all exact, this complex defines an exact functor

$$\Phi_s : K^b(\mathcal{O}_0) \longrightarrow K^b(\mathcal{O}_0),$$

by applying the complex of functors to a complex of modules and then taking the total complex of the resulting double complex.

There is another bounded complex of functors,

$$\Psi_s = \cdots \longrightarrow 0 \longrightarrow \mathrm{Id} \longrightarrow \theta_s \longrightarrow 0 \longrightarrow \cdots,$$

giving a functor $\Psi_s$ from $K^b(\mathcal{O}_0)$ to itself, where again $\theta_s$ is in degree zero, and this time the map from the identity functor to $\theta_s$ is the unit of the adjunction.

If $w = s_1 \cdots s_n$ is a reduced expression of an element of $W$, then we put $\Psi_w = \Psi_{s_1} \cdots \Psi_{s_n}$ and $\Phi_w = \Phi_{s_1} \cdots \Phi_{s_n}$. (This is well defined because the functors $\Phi_s$ and $\Psi_s$ satisfy the braid relations.)

LEMMA 2.2. *If $w_0$ is the longest element of $W$ and $w \in W$, then the complex $\Phi_w(\Delta(w_0))$ is a tilting resolution of $\Delta(w_0 w^{-1})$ and the complex $\Psi_w(\nabla(w_0))$ is a tilting resolution of $\nabla(w_0 w^{-1})$.*

*Proof.* We will prove only the first statement, the second is proved similarly. As $\Delta(w_0)$ is simple, and the duality $D$ fixes simple modules, we have that $\Delta(w_0)$ is tilting. As wall-crossing functors preserve tiltings (see, for example, [**7**, Proposition 11.1(e)]), the terms of the complex $\Phi_w(\Delta(w_0))$ are tilting. So we only need to prove that $\Phi_w(\Delta(w_0))$ is a resolution of $\Delta(w_0 w^{-1})$, or in other words, that

$$H^i(\Phi_w(\Delta(w_0))) = \delta_{i,0}\Delta(w_0 w^{-1}). \tag{3}$$

For every $w$ such that $ws > w$, we have the well-known exact sequence:

$$0 \longrightarrow \Delta(w) \longrightarrow \theta_s\Delta(w) \longrightarrow \Delta(ws) \longrightarrow 0.$$

This sequence combined with the fact that for every $x \in W$, we have $\theta_s\Delta(x) \cong \theta_s\Delta(xs)$, proves (3), by induction on the length of $w$. □

We have the following sequence of isomorphisms:

$$\begin{aligned} \mathrm{Ext}^i_{\mathcal{O}}(\Delta(w_0 w^{-1}), \nabla(w_0 v^{-1})) &\simeq \mathrm{Hom}_{K^b(\mathcal{O})}(\Phi_w(\Delta(w_0)), \Psi_v(\nabla(w_0))[i]) \\ &\simeq \mathrm{Hom}_{K^b(C)}(F_w \otimes \mathbb{C}, F^{-1}_{v^{-1}} \otimes \mathbb{C}[i]) \\ &\simeq \mathrm{Hom}_{K^b(R\otimes R)}(F_w, F^{-1}_{v^{-1}}[i]) \otimes_R \mathbb{C}. \end{aligned}$$

The first isomorphism follows from Lemma 2.2 and the fact that if $M$ and $N$ are tilting objects, then $\mathrm{Ext}^i_{\mathcal{O}}(M, N) = 0$ for all $i > 0$ (see, for example, [**7**, Proposition 11.1(f)]).

For the second isomorphism, we apply $\mathbb{V}$ and remark that on the one hand we have isomorphisms of functors from $K^b(\mathcal{O}_0)$ to $K^b(C)$: $\mathbb{V}\Phi_s \simeq F_s\mathbb{V}$ and $\mathbb{V}\Psi_s \simeq F_s^{-1}\mathbb{V}$ (see [**20**, Theorem 10]), and on the other $\mathbb{V}(\nabla(w_0)) \simeq \mathbb{V}(\Delta(w_0)) \simeq \mathbb{C}$ (which follows because

$\nabla(w_0) \cong \Delta(w_0)$ is simple). This together with the fact that $\mathbb{V}$ is fully faithful on tilting objects proves the second isomorphism. The third isomorphism is a consequence of [**21**, Proposition 8].

The graded Nakayama Lemma and (1) of Section 1 allows us to conclude the proof of the second case of Theorem 1.1, and the first case ($w = v$ and $i = 0$) follows easily by the adjunctions arguments as $F_s$ is right and left adjoint of $F_{s^{-1}}$ (see [**16**] or [**15**, Lemma 5.1 and Corollary 5.5]).

## 3. *Arbitrary Coxeter groups*

In this section, we prove Rouquier's formula for arbitrary Coxeter groups. This involves a more detailed look at Soergel bimodules, and in particular their behaviour under various natural subquotient functors.

### 3.1. *Filtration by support*

For $x \in W$, consider the reversed graph

$$\mathrm{Gr}(x) = \{(xv, v) \,|\, v \in V\} \subseteq V \times V,$$

and for any subset $A$ of $W$ consider the union of the corresponding graphs

$$\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x) \subseteq V \times V.$$

Given a finite set $A \subset W$, we can view $\mathrm{Gr}(A)$ as a subvariety of $V \times V$. If we identify $R \otimes R$ with the regular functions on $V \times V$, then $R_A$, the regular functions on $\mathrm{Gr}(A)$, is naturally a $\mathbb{Z}$-graded $R$-bimodule. If $A = \{x_1, \ldots, x_n\}$, then we will also write $R_A = R_{x_1,\ldots,x_n}$. For example, one may check that given $x \in W$ the bimodule $R_x$ has the following simple description: $R_x \cong R$ as a left module, and the right action is twisted by $x$: $m \cdot r = mx(r)$ for $m \in R_x$ and $r \in R$.

For any $R$-bimodule $M \in R$-Mod-$R$, we can view $M$ as an $R \otimes R$-module (because $R$ is commutative) and hence as a quasi-coherent sheaf on $V \times V$. Given any subset $A \subseteq W$ (not necessarily finite), we define

$$\Gamma_A M := \{m \in M \,|\, \operatorname{supp} m \subseteq \mathrm{Gr}(A)\}$$

to be the subbimodule consisting of elements whose support is contained in $\mathrm{Gr}(A)$. Note that $\Gamma_A$ is an endofunctor of $R$-Mod-$R$.

In the following, we will abuse notation and write $\leqslant x$ for the set $\{y \in W \,|\, y \leqslant x\}$ and similarly for $< x$, $\geqslant x$ and $> x$. With this notation, we obtain functors $\Gamma_{\leqslant x}$, $\Gamma_{<x}$, $\Gamma_{\geqslant x}$ and $\Gamma_{>x}$. For example, $\Gamma_{\leqslant x} = \Gamma_{\{y \in W \,|\, y \leqslant x\}}$.

In the sequel, an important role will be played by the 'subquotient functors':

$$\Gamma_{\geqslant x/>x} : M \longmapsto \Gamma_{\geqslant x} M / \Gamma_{>x} M,$$
$$\Gamma_{\leqslant x/<x} : M \longmapsto \Gamma_{\leqslant x} M / \Gamma_{<x} M.$$

Let $M \in R$-Mod-$R$ and assume that $M$ is finitely generated as an $R$-bimodule, and that the support of $M$ is contained in $\mathrm{Gr}(A)$ for some finite set $A \subseteq W$. We say that $M$ has a $\Delta$-*filtration* and write $M \in \mathcal{F}_\Delta$ if for all $x \in W$, $\Gamma_{\geqslant x/>x} M$ is isomorphic to a direct sum of shifts of $R_x$. Similarly, we say that $M$ has a $\nabla$-*filtration* and write $M \in \mathcal{F}_\nabla$ if for all $x \in W$, $\Gamma_{\leqslant x/<x} M$ is isomorphic to a direct sum of shifts of $R_x$. We call the full subcategories $\mathcal{F}_\Delta$ and $\mathcal{F}_\nabla$ of $R$-Mod-$R$ the categories of *bimodules with* $\Delta$-*flag* and *bimodules with* $\nabla$-*flag,* respectively.

A first important result about the categories $\mathcal{F}_\Delta$ and $\mathcal{F}_\nabla$ is Soergel's 'hin-und-her Lemma' [**22**, Lemma 6.3]. It states that given any enumeration $w_0, w_1, \ldots$ of the elements of $W$

compatible with the Bruhat order (that is, $w_i \leqslant w_j \Rightarrow i \leqslant j$) then $M \in R$-Mod-$R$ belongs to $\mathcal{F}_\nabla$ if and only if for all $i$ the subquotient

$$\Gamma_{\{w_0,\dots,w_i\}}M/\Gamma_{\{w_0,\dots,w_{i-1}\}}M$$

is isomorphic to a direct sum of shifts of $R_{w_i}$, in which case the natural map

$$\Gamma_{\leqslant w_i/<w_i}M \longrightarrow \Gamma_{\{w_0,\dots,w_i\}}M/\Gamma_{\{w_0,\dots,w_{i-1}\}}M \tag{4}$$

is an isomorphism. Similarly, $M \in \mathcal{F}_\Delta$ if and only if for all $i$,

$$\Gamma_{\{w_i,w_{i+1},\dots\}\cap A}M/\Gamma_{\{w_{i+1},\dots\}\cap A}M$$

are isomorphic to direct sums of shifts of $R_{w_i}$, in which case the natural map

$$\Gamma_{\geqslant w_i/>w_i}M \longrightarrow \Gamma_{\{w_i,w_{i+1},\dots\}\cap A}M/\Gamma_{\{w_{i+1},\dots\}\cap A}M \tag{5}$$

is an isomorphism (here $A \subseteq W$ denotes a finite set such that $\operatorname{supp} M \subseteq \operatorname{Gr}(A)$).

The following lemma (one of the first consequences of the hin-und-her Lemma) implies that if $M \in \mathcal{F}_\nabla$, then so are $\Gamma_{\leqslant x}M$ and $M/\Gamma_{\leqslant x}M$.

Lemma 3.1. *Let $M \in \mathcal{F}_\nabla$.*

(1) *The module $\Gamma_{\leqslant x/<x}(\Gamma_{\leqslant y}M)$ is zero unless $x \leqslant y$ in which case the natural map*

$$\Gamma_{\leqslant x/<x}(\Gamma_{\leqslant y}M) \longrightarrow \Gamma_{\leqslant x/<x}M \tag{6}$$

*is an isomorphism.*

(2) *The module $\Gamma_{\leqslant x/<x}(M/\Gamma_{\leqslant y}M)$ is zero unless $x \nleqslant y$ in which case the natural map*

$$\Gamma_{\leqslant x/<x}M \longrightarrow \Gamma_{\leqslant x/<x}(M/\Gamma_{\leqslant y}M) \tag{7}$$

*is an isomorphism.*

*Proof.* Statement (1) is straightforward. Let us prove (2). Choose an enumeration $w_0, w_1, \dots$ of the elements of $W$ compatible with the Bruhat order and such that $\{\leqslant y\} = \{w_0, w_1, \dots, w_k\}$ for some $k$. Let us abbreviate $\Gamma_{\leqslant i} = \Gamma_{\{w_0,\dots,w_i\}}$, $\Gamma_{<i} = \Gamma_{\{w_0,\dots,w_{i-1}\}}$ and

$$\Gamma_{\leqslant i/<i}(M) = \Gamma_{\leqslant i}(M)/\Gamma_{<i}(M).$$

By the hin-und-her lemma, we see that (2) is equivalent to the statement:

(2′) $\Gamma_{\leqslant i/<i}(M/\Gamma_{\leqslant k}M)$ is zero unless $i \nleqslant k$ in which case the natural map

$$\Gamma_{\leqslant i/<i}M \longrightarrow \Gamma_{\leqslant i/<i}(M/\Gamma_{\leqslant k}M) \tag{8}$$

is an isomorphism.

To prove (2′), consider the filtration

$$\cdots \subseteq F_i \subseteq F_{i+1} \subseteq \cdots$$

on $M/\Gamma_{\leqslant k}M$ obtained by taking the image of the filtration

$$\cdots \subseteq \Gamma_{\leqslant i}M \subseteq \Gamma_{\leqslant i+1}M \subseteq \cdots$$

on $M$. By the third isomorphism theorem

$$F_i/F_{i-1} \cong \begin{cases} 0 & \text{if } i \leqslant k, \\ \Gamma_{\leqslant i/<i}M & \text{otherwise.} \end{cases}$$

By the lemma below, we have $F_i = \Gamma_{\leqslant i}(M/\Gamma_{\leqslant k}M)$. Statement (2′) now follows. □

LEMMA 3.2. *Let $N \in R$-Mod-$R$ and fix an enumeration $w_0, w_1, \ldots$ of the elements of $W$ compatible with the Bruhat order. Suppose that we have a filtration $0 = N_{-1} \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$ of $N$ such that $N_i/N_{i-1}$ is isomorphic to a direct sum of shifts of $R_{w_i}$ for $0 \leqslant i \leqslant n$. Then $N_i = \Gamma_{\{w_0,w_1,\ldots,w_i\}}N$.*

*Proof.* Because $N_i$ is an extension of shifts of $R_{w_l}$ with $l \leqslant i$, we have $N_i \subseteq \Gamma_{\{w_0,\ldots,w_i\}}N$. It remains the prove the reverse inclusion. So choose $m \in \Gamma_{\{w_0,\ldots,w_i\}}N$ and let $k$ be minimal such that $m \in N_k$ but $m \notin N_{k-1}$. Then the image of $m$ in $N_k/N_{k-1}$ is non-zero. Using that $N_k/N_{k-1}$ is isomorphic to a direct sum of shifts of $R_{w_k}$, and that any non-zero element of $R_{w_k}$ has support equal to $\mathrm{Gr}(w_k)$ we see that $\mathrm{Gr}(w_k)$ is contained in the support of $m$. Hence, $w_k \in \{w_0, \ldots, w_i\}$, so $k \leqslant i$, so $m \in N_i$. Hence, $\Gamma_{\{w_0,\ldots,w_i\}}N \subseteq N_i$. □

The following lemma is the $\Delta$-version of Lemma 3.1. It implies that if $M \in \mathcal{F}_\Delta$, then so are $\Gamma_{\geqslant x}M$ and $M/\Gamma_{\geqslant x}M$ for any $x \in W$. The proof (which we omit) is similar to that of Lemma 3.1.

LEMMA 3.3. *Let $M \in \mathcal{F}_\Delta$.*

(1) *The module $\Gamma_{\geqslant x/>x}(\Gamma_{\geqslant y}M)$ is zero unless $x \geqslant y$ in which case the natural map*

$$\Gamma_{\geqslant x/>x}(\Gamma_{\geqslant y}M) \longrightarrow \Gamma_{\geqslant x/>x}M \tag{9}$$

*is an isomorphism.*

(2) *The module $\Gamma_{\geqslant x/>x}(M/\Gamma_{\geqslant y}M)$ is zero unless $x \ngeqslant y$ in which case the natural map*

$$\Gamma_{\geqslant x/>x}M \longrightarrow \Gamma_{\geqslant x/>x}(M/\Gamma_{\geqslant y}M) \tag{10}$$

*is an isomorphism.*

In [**22**, Section 5], Soergel proves that Soergel bimodules belong to both $\mathcal{F}_\Delta$ and $\mathcal{F}_\nabla$. Another important result is Soergel's Hom formula [**22**, Theorem 5.15]: for any $M \in \mathcal{F}_\Delta$ and $N \in \mathcal{B}$, or $M \in \mathcal{B}$ and $N \in \mathcal{F}_\nabla$, one has an isomorphism of graded right $R$-modules

$$\mathrm{Hom}(M, N) \cong \bigoplus_{x \in W} \mathrm{Hom}(\Gamma_{\geqslant x/>x}M, \Gamma_{\leqslant x/<x}N)(-2\ell(x)). \tag{11}$$

Finally, it is natural to ask how the support of a bimodule changes under the functor $M \mapsto MB_s$. It is straightforward (see [**24**, Lemma 4.14]) to show that if $M \in R$-Mod-$R$ has support contained in $\mathrm{Gr}(A)$ for some finite subset $A \subseteq W$, then we have

$$\mathrm{supp}(MB_s) \subseteq \mathrm{Gr}(A \cup As) \quad \text{and} \quad \mathrm{supp}(B_sM) \subseteq \mathrm{Gr}(A \cup sA).$$

It follows that if $s, t, \ldots, u \in \mathcal{S}$, then

$$\mathrm{supp}(B_sB_t \cdots B_u) \subseteq \mathrm{Gr}(\{\mathrm{id}, \mathrm{s}\}\{\mathrm{id}, \mathrm{t}\} \cdots \{\mathrm{id}, \mathrm{u}\}). \tag{12}$$

REMARK 3.4. This terminology $\Delta$-filtered and $\nabla$-filtered is intended to remind the reader of category $\mathcal{O}$: $\mathcal{F}_\Delta$ (respectively, $\mathcal{F}_\nabla$) can be thought of as those objects with standard (respectively, costandard) filtrations. The above results show that Soergel bimodules can be thought of as akin to tilting modules. As is well known (and follows from the vanishing formula (1) of Section 1), the functor of homomorphisms from (respectively, to) a tilting module is exact on complexes of costandard (respectively, standard) filtered objects. It is this analogy which motivates the introduction of $\Delta$- and $\nabla$-exact complexes in the next section.

3.2. Δ- *and* ∇-*exact complexes*

The following definition is fundamental to our proof of Rouquier's formula.

Definition 3.5. Let $A$ be a bounded complex of graded $R$-bimodules.

(1) $A$ is Δ-*exact* if $A^i \in \mathcal{F}_\Delta$ for all $i$ and, for all $x \in W$ the complex $\Gamma_{\geqslant x/>x}A$ is homotopic to zero.

(2) $A$ is ∇-*exact* if $A^i \in \mathcal{F}_\nabla$ for all $i$ and, for all $x \in W$ the complex $\Gamma_{\leqslant x/<x}A$ is homotopic to zero.

Remark 3.6. (1) Because an extension of acyclic complexes is acyclic, it follows that any Δ- or ∇-exact complex is acyclic.

(2) The canonical example of a Δ-exact complex is the 3-term complex

$$0 \longrightarrow \Gamma_{\geqslant y}M \longrightarrow M \longrightarrow M/\Gamma_{\geqslant y}M \longrightarrow 0, \tag{13}$$

for some $M \in \mathcal{F}_\Delta$. That this sequence is Δ-exact follows from Lemma 3.3.

(3) Similarly, if $M \in \mathcal{F}_\nabla$, then the sequence

$$0 \longrightarrow \Gamma_{\leqslant y}M \longrightarrow M \longrightarrow M/\Gamma_{\leqslant y}M \longrightarrow 0 \tag{14}$$

is ∇-exact. This follows from Lemma 3.1.

(4) If $A$ is Δ-exact, then so are $\Gamma_{\geqslant x}A$ and $A/\Gamma_{\geqslant x}A$ for any $x \in W$. Similarly, if $A$ is ∇-exact, then so are $\Gamma_{\leqslant x}A$ and $A/\Gamma_{\leqslant x}A$ for any $x \in W$. Again these statements follow easily from Lemmas 3.1 and 3.3.

(5) If $A$ is a bounded complex of modules belonging to $\mathcal{F}_\Delta$, then $\Gamma_{\geqslant x/>x}A^i$ is isomorphic to a direct sum of shifts of $R_x$ for all $x \in W$ and $i \in \mathbb{Z}$. One may show that a bounded complex of modules all of whose terms are isomorphic to direct sums of shifts of $R_x$ is homotopic to zero if and only if it is acyclic. (Such a complex of $R$-bimodules is homotopic to zero if and only if it is homotopic to zero as a complex of left $R$-modules. However, each $R_x$ is projective as a left $R$-module, and so any acyclic complex splits.) Hence, we could have replaced 'homotopic to zero' by 'exact' in the above definition. This also explains the origin of the terminology.

Example 3.7. Consider the complexes

$$\widetilde{F_s} = \cdots \longrightarrow R_s(-1) \xrightarrow{\eta'_s} B_s \xrightarrow{m_s} R(1) \longrightarrow \cdots,$$
$$\widetilde{E_s} = \cdots \longrightarrow R(-1) \xrightarrow{\eta_s} B_s \xrightarrow{m'_s} R_s(1) \longrightarrow \cdots,$$

where $m_s$ and $\eta_s$ are as in Subsection 2.3 and $\eta'_s$ and $m'_s$ are determined by $\eta'_s(1) = \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$ and $m'_s(f \otimes g) = fs(g)$, respectively. We claim that $\widetilde{F_s}$ is Δ-exact, but that $\widetilde{E_s}$ is not. Indeed, one has isomorphisms

$$\Gamma_{\geqslant s}\widetilde{F_s} \cong \cdots \longrightarrow R_s(-1) \xrightarrow{\mathrm{id}} R_s(-1) \longrightarrow 0 \longrightarrow \cdots,$$
$$\Gamma_{\geqslant s}\widetilde{E_s} \cong \cdots \longrightarrow 0 \longrightarrow R_s(-1) \xrightarrow{\alpha_s\cdot} R_s(1) \longrightarrow \cdots$$

and

$$\Gamma_{\geqslant \mathrm{id}/\geqslant s}\widetilde{F_s} \cong \cdots \longrightarrow 0 \longrightarrow R(1) \xrightarrow{\mathrm{id}} R(1) \longrightarrow \cdots,$$
$$\Gamma_{\geqslant \mathrm{id}/\geqslant s}\widetilde{E_s} \cong \cdots \longrightarrow R(-1) \xrightarrow{\alpha_s\cdot} R(1) \longrightarrow 0 \longrightarrow \cdots.$$

A dual calculation shows that $\widetilde{E_s}$ is ∇-exact and that $\widetilde{F_s}$ is not.

In fact, $\widetilde{F_s}$ and $\widetilde{E_s}$ are examples of augmented Rouquier complexes which will be introduced in Subsection 3.4. The above calculations give a concrete example of Proposition 3.12.

The following lemma is the central technical tool of this paper. It shows that $\Delta$- (respectively, $\nabla$-) exact complexes are acyclic for the functor of homomorphisms to (respectively, from) a Soergel bimodule.

Proposition 3.8. *Let $A$ denote a bounded complex of $R$-bimodules and let $B \in \mathcal{B}$ be a Soergel bimodule*:

(1) *if $A$ is $\Delta$-exact, then* $\mathrm{Hom}_K(A, B) = 0$;
(2) *if $A$ is $\nabla$-exact, then* $\mathrm{Hom}_K(B, A) = 0$.

*Proof.* We only prove (1). Statement (2) is proved by a very similar argument.

Suppose first that $A$ is a $\Delta$-exact complex consisting of only three non-zero terms:

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow 0.$$

Let us apply $\mathrm{Hom}(-, B)$. Because $\mathrm{Hom}(-, B)$ is left exact, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M^3, B) \longrightarrow \mathrm{Hom}(M^2, B) \xrightarrow{r} \mathrm{Hom}(M^1, B). \tag{15}$$

We claim that $r$ is in fact surjective. Let us consider (15) in each degree separately. So fix a degree $i \in \mathbb{Z}$. Certainly, we have an exact sequence of finite-dimensional vector spaces

$$0 \longrightarrow \mathrm{Hom}(M^3, B)_i \longrightarrow \mathrm{Hom}(M^2, B)_i \xrightarrow{r_i} \mathrm{Hom}(M^1, B)_i.$$

On the other hand, by the fact that our sequence is $\Delta$-exact we have, for any $x \in W$,

$$\Gamma_{\geqslant x/>x} M^2 \cong \Gamma_{\geqslant x/>x} M^1 \oplus \Gamma_{\geqslant x/>x} M^3.$$

Soergel's Hom formula (11) gives:

$$\dim \mathrm{Hom}(M^3, B)_i + \dim \mathrm{Hom}(M^1, B)_i = \dim \mathrm{Hom}(M^2, B)_i.$$

We conclude that $r_i$ is surjective. Hence, $r$ is surjective as claimed and indeed we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M^3, B) \longrightarrow \mathrm{Hom}(M^2, B) \longrightarrow \mathrm{Hom}(M^1, B) \longrightarrow 0.$$

We now turn to the general case and argue by induction on the size of the set

$$\mathrm{supp}\, A := \{x \in W \mid \text{there exists an } i \text{ such that } \Gamma_{\geqslant x/>x} A^i \neq 0\}.$$

If $|\mathrm{supp}\, A| = 1$, then $A$ is homotopic to zero (as follows directly from the definition of $\Delta$-exactness) and the lemma holds in this case.

For the general case, fix $x \in \mathrm{supp}\, A$ which is maximal in the Bruhat order and consider the sequence of $\Delta$-exact complexes

$$0 \longrightarrow \Gamma_{\geqslant x} A \longrightarrow A \longrightarrow A/\Gamma_{\geqslant x} A \longrightarrow 0.$$

By the remarks following the definition of $\Delta$-exact each row and column of this sequence is $\Delta$-exact.

By the 3-term case considered above, applying $\mathrm{Hom}(-, B)$ yields an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}^\bullet(A/\Gamma_{\geqslant x} A, B) \longrightarrow \mathrm{Hom}^\bullet(A, B) \longrightarrow \mathrm{Hom}^\bullet(\Gamma_{\geqslant x} A, B) \longrightarrow 0.$$

Now $\mathrm{supp}(\Gamma_{\geqslant x} A) = \{x\}$ and $|\mathrm{supp}(A/\Gamma_{\geqslant x} A)| < |\mathrm{supp}\, A|$. Hence, we can apply induction to conclude that the complexes $\mathrm{Hom}^\bullet(A/\Gamma_{\geqslant x} A, B)$ and $\mathrm{Hom}^\bullet(\Gamma_{\geqslant x} A, B)$ are acyclic. Hence,

$\mathrm{Hom}^\bullet(A,B)$ is acyclic too, by the long exact sequence of cohomology. Hence, $\mathrm{Hom}_K(A,B) = H^0(\mathrm{Hom}^\bullet(A,B)) = 0$ as claimed. □

Corollary 3.9. *If $F \in K^b(\mathcal{F}_\Delta)$ is $\Delta$-exact and $G \in K^b(\mathcal{B})$ or $F \in K^b(\mathcal{B})$ and $G \in K^b(\mathcal{F}_\nabla)$ is $\nabla$-exact, then we have $\mathrm{Hom}_K(F,G) = 0$.*

*Proof.* We handle the case of $F \in K^b(\mathcal{F}_\Delta)$ and $G \in K^b(\mathcal{B})$. The dual case is analogous.

We have seen in Proposition 3.8 above that if $F$ is $\Delta$-exact, then $\mathrm{Hom}_K(F,B) = 0$ for any Soergel bimodule $B \in \mathcal{B}$. We prove the proposition by induction on $\ell(G) := |\{i \in \mathbb{Z} \mid G^i \neq 0\}|$. The case $\ell(G) = 0$ is trivial and the case $\ell(G) = 1$ follows by the above proposition.

So fix $G \in K^b(\mathcal{B})$ and assume that we have proved the lemma for all complexes of Soergel bimodules $G'$ with $\ell(G') < \ell(G)$. Choose $i$ maximal with $G^i \neq 0$. We have a distinguished triangle (see (2))

$$w_{\geqslant i}G \longrightarrow G \longrightarrow w_{<i}G \xrightarrow{[1]}, \tag{16}$$

with $\ell(w_{\geqslant i}G) = 1$ and $\ell(w_{<i}G) = \ell(G) - 1$. As $\mathrm{hom}(F,-)$ is cohomological, so is $\mathrm{Hom}(F,-)$. If we apply $\mathrm{Hom}(F,-)$ to (16), then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(F, w_{\geqslant i}G) \longrightarrow \mathrm{Hom}(F,G) \longrightarrow \mathrm{Hom}(F, w_{<i}G) \longrightarrow \cdots,$$

and induction allows us to conclude that $\mathrm{Hom}(F,G) = 0$ as claimed. □

## 3.3. *Exactness properties of Rouquier complexes*

In this section, we prove the complexes $F_x$ (respectively, $E_x$) for $x \in W$ are almost $\Delta-$ (respectively, $\nabla-$) exact. More precisely, the goal is to prove the following proposition.

Proposition 3.10. *For $x, y \in W$, we have isomorphisms in the homotopy category*

$$\Gamma_{\leqslant y/<y} E_x \cong \begin{cases} R_x(\ell(x)) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

*Similarly,*

$$\Gamma_{\geqslant y/>y} F_x \cong \begin{cases} R_x(-\ell(x)) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We prove the first isomorphism, the second follows by a dual argument. We now prove this isomorphism by induction on $\ell(x)$. It can be verified by hand for $\ell(x) = 0, 1$. So fix $x$ and choose $s \in S$ with $xs < x$ so that $E_x = E_{xs}E_s$. By induction, we may assume that the first isomorphism in the proposition holds for $E_{xs}$. Our aim is to show that it also holds for $E_x$.

Let us choose an enumeration $w_0, w_1, w_2, \ldots$ of the elements of $W$ compatible with the Bruhat order and such that $w_{m+1} = w_m s$ for all even $m$. Such an enumeration can be constructed by first choosing an enumeration of $W/\langle s\rangle$ compatible with the Bruhat order and then refining it to an enumeration of $W$. It follows from [**24**, Proposition 6.5] that for any Soergel bimodule $B$ the natural map gives an isomorphism:

$$(\Gamma_{\leqslant m+1/<m} B) B_s \xrightarrow{\sim} (\Gamma_{\leqslant m+1/<m} B B_s) \quad \text{for } m \text{ even.}$$

(We use the same notation as in the proof of Lemma 3.1.) Hence, for any complex $F \in K^b(\mathcal{B})$ of Soergel bimodules we have an isomorphism:

$$(\Gamma_{\leqslant m+1/<m} F) E_s \xrightarrow{\sim} (\Gamma_{\leqslant m+1/<m} F E_s) \quad \text{for } m \text{ even.} \tag{17}$$

We first deal with the case of $y \in \{x, xs\}$. Fix $n$ such that $w_{n+1} = x$. Then $n$ is necessarily even and $w_n = xs$. If we apply induction and (17), then we obtain

$$\Gamma_{\leqslant n+1/<n}(E_x) \cong \Gamma_{\leqslant n+1/<n}(E_{xs})E_s \cong \cdots \longrightarrow R_{xs}(\ell(x)-2) \longrightarrow R_{x,xs}(\ell(x)) \longrightarrow \cdots,$$

where $R_{x,xs}(\ell(x))$ occurs in complex degree 0 and all terms which are not displayed are zero. Indeed, by induction we have $\Gamma_{\leqslant n+1/<n}(E_{xs}) \cong R_{xs}(\ell(xs))$ and $R_{xs}B_s \cong R_{xs,x}(1)$.

The result in this case then follows by applying $\Gamma_{\leqslant n+1/<n+1}$ and $\Gamma_{\leqslant n/<n}$ and using the isomorphisms of functors on $K^b(\mathcal{F}_\nabla)$ (valid for any $i$):

$$\Gamma_{\leqslant i/<i}(-) \cong \Gamma_{\leqslant i}(\Gamma_{\leqslant i+1/<i}(-)), \tag{18}$$

$$\Gamma_{\leqslant i+1/<i+1}(-) \cong \Gamma_{\leqslant i+1/<i+1}(\Gamma_{\leqslant i+1/<i}(-)). \tag{19}$$

(Both of these isomorphisms follow from Lemma 5.)

We now deal with the case $y \notin \{x, xs\}$. We may assume $ys < y$ and fix $p$ (again even) such that $y_p = ys$ and $y_{p+1} = y$. As above, we have

$$\Gamma_{\leqslant p+1/<p}(E_x) = \Gamma_{\leqslant p+1/<p}(E_{xs}E_s) \cong \Gamma_{\leqslant p+1/<p}(E_{xs})E_s.$$

Consider the exact sequence of complexes of bimodules

$$0 \longrightarrow \Gamma_{\leqslant p/<p}E_{xs} \longrightarrow \Gamma_{\leqslant p+1/<p}E_{xs} \longrightarrow \Gamma_{\leqslant p+1/<p+1}E_{xs} \longrightarrow 0.$$

Each term on the left (respectively, right) is isomorphic to a direct sum of shifts of $R_{ys}$ (respectively, $R_y$). By [**22**, Lemma 5.8], any such extension splits when we restrict to $R$-Mod-$R^s$. Hence, we have an exact sequence of complexes

$$0 \longrightarrow (\Gamma_{\leqslant p/<p}E_{xs})B_s \longrightarrow (\Gamma_{\leqslant p+1/<p}E_{xs})B_s \longrightarrow (\Gamma_{\leqslant p+1/<p+1}E_{xs})B_s \longrightarrow 0,$$

where each row is split exact. Hence, we have an exact triangle in $K^b(R\text{-Mod-}R)$

$$(\Gamma_{\leqslant p/<p}E_{xs})B_s \longrightarrow \Gamma_{\leqslant p+1/<p}(E_{xs}B_s) \longrightarrow (\Gamma_{\leqslant p+1/<p+1}E_{xs})B_s \xrightarrow{[1]}.$$

By induction, the left- and right-hand terms are homotopic to zero. We conclude that $\Gamma_{\leqslant p+1/<p}(E_{xs}B_s)$ is also homotopic to zero and hence

$$\Gamma_{\leqslant p+1/<p+1}(E_{xs}B_s) \cong \Gamma_{\leqslant p/<p}(E_{xs}B_s) \cong 0,$$

again using (18) and (19).

Finally, we have a distinguished triangle

$$E_{xs}(-1) \longrightarrow E_{xs}B_s \longrightarrow E_{xs}E_s \xrightarrow{[1]},$$

and applying $\Gamma_{\leqslant p+1/<p+1}(-)$ and $\Gamma_{\leqslant p/<p}(-)$ the two left-hand terms are zero. Hence,

$$\Gamma_{\leqslant p+1/<p+1}(E_x) \cong \Gamma_{\leqslant p/<p}(E_x) \cong 0,$$

which is what we wanted to show. □

### 3.4. *Augmented Rouquier complexes*

Given any braid $\sigma = s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n} \in B_W$, let $\epsilon(\sigma) = \sum_{i=1}^n m_i$.

The following is standard.

LEMMA 3.11. *Let $\sigma \in B_W$ and let $w$ denote its image in $W$. We have*

$$H^i(F_\sigma) = \begin{cases} R_w(-\epsilon(\sigma)) & \textit{if } i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* It is well known that Soergel bimodules are free when regarded as left or right $R$-modules. (Any Bott–Samelson bimodule $B_sB_t\cdots B_u$ is free as a left or right $R$-bimodule and hence so is any direct summand.) It follows that the functor of tensoring on the right or left by a Soergel bimodule is exact.

We prove the lemma by induction on the length $\ell(\sigma)$ of a minimal word for $\sigma$, with the cases $\ell(\sigma)=0,1$ following by direct calculation from the definition of $F_\sigma$. First assume $\sigma=\sigma' s$ for some $\sigma'\in B_W$ with $\ell(\sigma')=\ell(\sigma)-1$. The cohomology of $F_\sigma$ is the cohomology of the double complex $F_{\sigma'}B_s\to F_{\sigma'}R(1)$. However, by induction and the above remarks, we get that the cohomology of $F_\sigma$ is the cohomology of the complex $R_wB_s(-\epsilon(\sigma'))\to R_wR(-\epsilon(\sigma')+1)$. Now direct calculation shows that this complex is quasi-isomorphic to a complex with only non-zero term $R_{ws}[-\epsilon(\sigma)-1]$ in degree zero, which establishes the induction step in this case.

The case when $\sigma=\sigma' s^{-1}$ for some $s\in\mathcal{S}$ is handled in a similar manner. □

Now fix $w\in W$. The above lemma shows that the cohomology of $F_w$ (respectively, $E_w$) is concentrated in degree zero, where it is isomorphic to $R_w(-\ell(w))$ (respectively, $R_w(\ell(w))$). From the definitions, it is clear that the non-zero terms of the complex $F_w$ (respectively, $E_w$) are in degrees $\geqslant 0$ (respectively, in degrees $\leqslant 0$). Hence, we have truncation morphisms

$$f_w : R_w(-\ell(w)) = H^0(F_w) = \tau_{\leqslant 0}F_w \longrightarrow F_w$$

and

$$e_w : E_w \longrightarrow \tau_{\geqslant 0}E_w = H^0(E_w) = R_w(\ell(w)).$$

We set

$$\widetilde{F_w} := \operatorname{cone}(f_w), \tag{20}$$

$$\widetilde{E_w} := \operatorname{cone}(e_w). \tag{21}$$

We call these complexes *augmented Rouquier complexes*.

As $F_w$ and $E_w$ only have cohomology in degree zero, the augmented Rouquier complexes $\widetilde{F_w}$ and $\widetilde{E_w}$ are exact. In fact, they are exact in a much stronger sense. By Proposition 3.10, we have the following corollary.

COROLLARY 3.12. *The complex $\widetilde{F_w}$ is $\Delta$-exact and the complex $\widetilde{E_w}$ is $\nabla$-exact.*

Combining this result with Corollary 3.9 yields the following corollary.

COROLLARY 3.13. *For $w,v\in W$ and $H\in K^b(\mathcal{B})$ and $m\in\mathbb{Z}$, we have*

$$\operatorname{Hom}_K(H,\widetilde{E_v}[m]) = 0 = \operatorname{Hom}_K(\widetilde{F_w},H[m]).$$

## 3.5. *Proof of the main theorem*

In this final section, we will see how $\Delta$-exactness and $\nabla$-exactness enters the story of the 2-braid group.

*Proof of Theorem* 1.1. We have the distinguished triangles in the triangulated category $K=K^b(R\text{-Mod-}R)$:

$$R_w(-\ell(w)) \longrightarrow F_w \longrightarrow \widetilde{F_w} \xrightarrow{[1]}, \tag{22}$$

and for any integer $m$:

$$E_v[m] \longrightarrow R_v(\ell(v))[m] \longrightarrow \widetilde{E_v}[m] \xrightarrow{[1]} . \tag{23}$$

By applying the cohomological functor $\mathrm{Hom}_K(-, E_v[m])$ to the triangle (22), we obtain, using Corollary 3.13:

$$\mathrm{Hom}_K(F_w, E_v[m]) \cong \mathrm{Hom}_K(R_w(-\ell(w)), E_v[m]). \tag{24}$$

Similarly, applying $\mathrm{Hom}_K(F_w, -)$ to (23) yields an isomorphism

$$\mathrm{Hom}_K(F_w, E_v[m]) \cong \mathrm{Hom}_K(F_w, R_v(\ell(v))[m]). \tag{25}$$

If $w = v$, then from (12) we have $\operatorname{supp} E_v^i \subset \mathrm{Gr}(< v)$ for $i < 0$ and $\operatorname{supp} E_v^0 = \mathrm{Gr}(\leqslant v)$. Hence, $\mathrm{Hom}(R_v(-\ell(w)), E_v^i) = 0$ for $i \neq 0$ and (as graded $R$-modules)

$$\mathrm{Hom}(R_v(-\ell(w)), E_v^0) = \mathrm{Hom}(R_v(-\ell(v)), R_v(\ell(v))(-2\ell(v))) = R,$$

by Soergel's Hom formula (11). Hence, the complex $\mathrm{Hom}^\bullet(R_v(-\ell(v)), E_v)$ is concentrated in degree zero, where it is isomorphic to $R$. Hence, by (24),

$$\mathrm{Hom}(F_v, E_v[m]) = \begin{cases} R & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume that $w \neq v$. We will prove that

$$\mathrm{Hom}_K(F_w, E_v[m]) = 0,$$

by considering two cases.

For the first case, assume that $w \nleqslant v$. Using (24), we have to show that

$$\mathrm{Hom}_K(R_w(-\ell(w)), E_v[m]) = 0.$$

For all $i \in \mathbb{Z}$, $E_v^i$ is a Soergel bimodule and $\operatorname{supp} E_v^i \subseteq \mathrm{Gr}(\leqslant v)$ (see (12)). It follows that $E_v^i$ has a filtration with subquotients isomorphic to direct sums of shifts of $R_x$ with $x \leqslant v$. Now, using the fact that $\mathrm{Hom}(R_x, R_y) = 0$ if $x \neq y$, then we conclude that $\mathrm{Hom}(R_w, E_v^i) = 0$ (or alternatively one may use Soergel's Hom formula). The desired vanishing then follows.

For the second case, assume that $w < v$ so that $v \nleqslant w$. Using (25), we have to prove that

$$\mathrm{Hom}_K(F_w, R_v(\ell(v))[m]) = 0.$$

By a similar argument to that of the previous paragraph, we see that

$$\mathrm{Hom}(F_w^i, R_v) = 0 \quad \text{for all } i.$$

The theorem now follows. □

*Acknowledgements.* We would like to thank Wolfgang Soergel who explained how useful it is to know that a complex of Soergel bimodules is Δ-exact. We also benefited from discussions with Raphaël Rouquier. We would like to thank Catharina Stroppel, Thorge Jensen and the referee for useful comments. Finally, both authors would like to thank the MPI for a productive research environment.

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