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Pre-canonical bases on affine Hecke algebras

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A B S T R A C T

For any affine Weyl group, we introduce the pre-canonical bases. They are a set of bases $\{\mathbf{N}^i\}_{1\leq i\leq m+1}$ (where m is the height of the highest root) of the spherical Hecke algebra that interpolates between the standard basis $\mathbf{N}^1$ and the canonical basis $\mathbf{N}^{m+1}$. The expansion of $\mathbf{N}^{i+1}$ in terms of the $\mathbf{N}^i$ is in many cases very simple and we conjecture that in type A it is positive.

1. Introduction

This paper introduces the notion of pre-canonical bases on the spherical Hecke algebra. The motivation comes from the study of Kazhdan-Lusztig polynomials in representation theory, and especially from Schützer's work on character formulas for Lie groups. From a computational point of view, the most interesting feature of the definition is that

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these bases interpolate between the standard and the canonical bases, thus dividing the hard problem of calculating Kazhdan-Lusztig polynomials (or q-analogues of weight multiplicities) into a finite number of much easier problems.

1.1. Overlook

Before going any further let us roughly describe a simple example of the pre-canonical bases on a spherical Hecke algebra. Let Φ be a root system of type A_3 with simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$. Let X^+ be the set of dominant weights and let $\widetilde{\mathcal{H}}$ be the spherical Hecke algebra with scalars $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The definition of $\widetilde{\mathcal{H}}$ will come later; for now we just need to know that $\widetilde{\mathcal{H}}$ has a standard basis $\{\mathbf{H}_\lambda : \lambda \in X^+\}$ and a canonical (or Kazhdan-Lusztig) basis $\{\underline{\mathbf{H}}_\lambda : \lambda \in X^+\}$. In this case there are four pre-canonical bases, namely

$$\{\mathbf{N}^i_\lambda : \lambda \in X^+\} \quad \text{for } 1 \leq i \leq 4,$$

with $\mathbf{N}^4_\lambda$ being the canonical basis $\underline{\mathbf{H}}_\lambda$ and $\mathbf{N}^1_\lambda$ being the standard basis $\mathbf{H}_\lambda$. Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 \in X^+$ where $\{\varpi_1, \varpi_2, \varpi_3\}$ are the fundamental weights. The three "simple" decompositions mentioned before come in very different flavors.

We use the notation $\alpha_{13} := \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_{12} := \alpha_1 + \alpha_2$ and $\alpha_{23} := \alpha_2 + \alpha_3$. The first decomposition is rather simple.

$$\mathbf{N}^4_\lambda = \sum_{k=0}^{\min(a,c)} q^k \mathbf{N}^3_{\lambda - k\alpha_{13}}. \tag{1.1}$$

This type of decomposition is a special case of a general phenomenon in type A_n: in fact, a similar behavior occurs in the decomposition of $\mathbf{N}^{i+1}$ in terms of $\mathbf{N}^i$ when $i \geq n/2 + 1$ (see Theorem 1.4).

For the second decomposition we need to introduce the set I_λ. It is the set of $\mu \in X^+$ such that there exist $n, m, l \in \mathbb{N}$ with

$$\mu = \lambda - n\alpha_{12} - m\alpha_{23} \quad \text{(in this case we consider } l \text{ to be 0)}$$

or

$$\lambda - n\alpha_{12} - m\alpha_{23} \in \mathbb{N}\varpi_1 + \mathbb{N}\varpi_3 \text{ and } \mu = \lambda - n\alpha_{12} - m\alpha_{23} - l\alpha_{13} \in X^+.$$

For $\mu \in I_\lambda$ we define $d(\mu) := n + m + 2l$.

$$\mathbf{N}^3_\lambda = \sum_{\mu \in I_\lambda} q^{d(\mu)} \mathbf{N}^2_\mu. \tag{1.2}$$

The last decomposition is given by the formula

$$\mathbf{N}^2_\lambda = \sum_{\substack{\mu \in X^+ \\ \mu \leq \lambda}} q^{\mathrm{ht}(\lambda-\mu)} \mathbf{N}^1_\mu, \tag{1.3}$$

where $\leq$ is the usual order on weights (i.e. $\mu \leq \lambda$ means that $\lambda - \mu \in \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2 + \mathbb{N}\alpha_3$) and ht denotes the height of a weight.

The basis $\mathbf{N}^2_\lambda$ would be the canonical basis "if all Kazhdan-Lusztig polynomials were trivial." Geometrically, $\mathbf{N}^2_\lambda$ is the character of the constant sheaf on the corresponding Schubert variety. Equation (1.3) will remain valid for any root system.

1.2. Pre-canonical bases

Let us fix some notation before we can introduce the pre-canonical bases. Let Φ be a root system and $X \supset \Phi$ be a corresponding weight lattice. We fix a system of positive roots $\Phi^+ \subset \Phi$ and let Δ be the corresponding set of simple roots. Let ρ be the half-sum of the positive roots. For an integer $i \geq 1$ define $\Phi^{\geq i}$ to be the set of positive roots with height at least i, or in formulas

$$\Phi^{\geq i} := \{\alpha \in \Phi^+ : \mathrm{ht}(\alpha) \geq i\}.$$

Let W_f be the finite Weyl group attached to Φ. We say that a weight $\lambda \in X$ is *regular* if there is no reflection $s \in W_f$ which fixes $\lambda + \rho$. For $\lambda \in X$ regular we define $w_\lambda \in W_f$ to be the unique element such that $w_\lambda \cdot \lambda$ is dominant (here $\cdot$ stands for the dot action, defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$).

Let $W_a = W_f \ltimes \mathbb{Z}\Phi$ be the corresponding affine Weyl group. Let $\mathcal{H}$ be the Hecke algebra of W_a over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ with standard basis $\{\mathbf{H}_x\}$ and Kazhdan-Lusztig basis $\{\underline{\mathbf{H}}_x\}$. For $\lambda \in \mathbb{Z}\Phi \subset X$, we think of the translation t_λ as an element of W_a. If $\lambda \in \mathbb{Z}\Phi \cap X^+$, let $\theta(\lambda) = t_\lambda w_0$ and $\underline{\mathbf{H}}_\lambda := \underline{\mathbf{H}}_{\theta(\lambda)} \in \mathcal{H}$. Then, for $\lambda \in \mathbb{Z}\Phi$, we define

$$\widetilde{\underline{\mathbf{H}}}_\lambda = \begin{cases} (-1)^{\ell(w_\lambda)} \underline{\mathbf{H}}_{w_\lambda \cdot \lambda}, & \text{if } \lambda \text{ is regular;} \\ 0, & \text{if } \lambda \text{ is singular.} \end{cases}$$

(See Section 2.1 for how to extend the definition of $\widetilde{\underline{\mathbf{H}}}_\lambda$ to any $\lambda \in X$.)

Definition 1.1 *(Pre-canonical bases).* For $i \geq 2$ and $\lambda \in X^+$ define

$$\mathbf{N}^i_\lambda := \sum_{I \subset \Phi^{\geq i}} (-q)^{|I|} \widetilde{\underline{\mathbf{H}}}_{\lambda - \sum_{\alpha \in I} \alpha}. \tag{1.4}$$

For $i = 1$ the definition is almost the same as eq. (1.4), only that one has to normalize by some scalar (for details see Definition 2.12). For any fixed $i \geq 1$, the set $\mathbf{N}^i := \{\mathbf{N}^i_\lambda :$

$\lambda \in X^+\}$ is called the i^{th} *pre-canonical basis*. It is a basis of the *spherical Hecke algebra* $\widetilde{\mathcal{H}}$, the decategorification of any of the two categories appearing in the Geometric Satake equivalence (see eq. (2.2) for an easy combinatorial definition of $\widetilde{\mathcal{H}}$ and eq. (2.3) for the definition of its standard basis $\{\mathbf{H}_\lambda\}$). The second part of the following theorem is a q-deformation of the main result of Schützer's paper [17] and it is at the origin of the definition of the pre-canonical bases.

Theorem 1.2. *For $\lambda \in X^+$ we have the following equations*

$$\mathbf{N}^1_\lambda = \mathbf{H}_\lambda \qquad \textit{and} \qquad \mathbf{N}^2_\lambda = \sum_{\substack{\mu \in X^+ \\ \mu \leq \lambda}} q^{\mathrm{ht}(\lambda-\mu)} \mathbf{N}^1_\mu .$$

The fact that $\mathbf{N}^{m+1}_\lambda = \underline{\mathbf{H}}_\lambda$ if m is the height of the highest root follows directly the definition of the pre-canonical bases. We remark that equality $\mathbf{N}^1_\lambda = \mathbf{H}_\lambda$ gives a closed formula for all inverse (spherical) Kazhdan-Lusztig polynomials for all affine Weyl groups.

Remark 1.3. The spherical Hecke algebra is isomorphic via the *Satake transform* to the algebra of symmetric functions $\mathbb{Z}[v, v^{-1}][X]^{W_f}$ (cf. Remark 2.4). Under this isomorphism, the Kazhdan-Lusztig basis $\{\underline{\mathbf{H}}_\lambda\}$ corresponds to the Weyl characters and the standard basis $\{\mathbf{H}_\lambda\}$ to the Hall-Littlewood polynomials (see e.g. [19]). It follows that, after applying Satake, the pre-canonical bases also yield new bases of the ring of symmetric functions which interpolate between Weyl characters and Hall-Littlewood polynomials.

1.3. Main conjecture

Suppose now that Φ is a root system of type A_n. For an integer $1 \leq i \leq n$, let Φ^i be the set of positive roots of height i. For $\lambda, \mu \in X^+$, we write $\mu \leq_i \lambda$ if $\lambda - \mu$ can be written as a positive integral linear combination of elements of Φ^i.

Theorem 1.4. *Let $n/2 + 1 \leq i \leq n$. For $\lambda \in X^+$ we have*

$$\mathbf{N}^{i+1}_\lambda = \sum_{\mu \leq_i \lambda} q^{\frac{1}{i}\,\mathrm{ht}(\lambda-\mu)} \mathbf{N}^i_\mu .$$

The formula in Theorem 1.4 is incredibly simple, and although we hope that a formula for all i will be found in the future (cf. Remark 1.8), we do not expect it to be as simple as that. In fact, we expect a different behavior for small i much more in the vein of eq. (1.3), as the following theorem illustrates.

Theorem 1.5. *The formulas for all the decompositions of $\mathbf{N}^{i+1}$ in terms of $\mathbf{N}^i$ in type A_3 are the ones explained in Section 1.1. The corresponding formulas in type A_4 are the ones explained in Section 5.2.*

We remark that the corresponding formulas in type A_2 were found by the first two authors of this paper in [11] and in type A_1 they are trivial. The following is the central conjecture of this paper.

Conjecture 1.6. *If Φ is a root system of type A_n, for each $i \geq 0$ we have*

$$\mathbf{N}_\lambda^{i+1} \in \sum_{\mu \in X^+} \mathbb{N}[q]\ \mathbf{N}_\mu^i$$

Remark 1.7. The only non trivial part of the conjecture is the positivity statement. We have the following evidence to believe in the validity of this conjecture. By Theorem 1.2 the conjecture is verified for $i = 1$ and all n. By Theorem 1.4 it is also verified for $n/2 + 1 \leq i$. By Theorem 1.5 the conjecture is verified for all i when $n \leq 4$. It was proved by Shimozono [18] following earlier work by Lascoux [9] that in type A_n the decomposition of $\mathbf{N}_\lambda^{n+1}$ in terms of the $\mathbf{N}^2$ basis (that using Theorem 1.2 one proves that it is the so-called *atomic decomposition*) is positive. Finally, we have checked this conjecture in several hundred cases by computer in types A_5 and A_6 using SageMath [16] with the help of the code developed in [4].

Remark 1.8. Conjecture 1.6, as written, is false for a general root system. In type D_4 a counter-example [10, Example 2.6] shows that $\underline{\mathbf{H}}_\lambda$ is not positive in general when decomposed in terms of $\mathbf{N}^2$. However, as Lecouvey and Lenart explain, the failure of this positivity seems to be mild and it would be interesting to determine whether there exists a “stable range” where the conjecture holds.

Remark 1.9. The name “pre-canonical bases” is inspired by this conjecture. We see Lusztig generational philosophy as a conjectural set of “post-canonical bases”, this time interpolating between the canonical basis and the p-canonical basis.

Remark 1.10. Empirical data in ranks ≤ 8 suggest that it should be possible to find a combinatorial formula for the polynomials appearing in the right-hand side of Conjecture 1.6. More precisely, we believe that for $i > 1$ the polynomials $P_i(\lambda, \mu) \in \mathbb{N}[q]$ defined by the formula

$$\mathbf{N}_\lambda^{i+1} = \sum_{\mu \in X^+} P_i(\lambda, \mu)\ \mathbf{N}_\mu^i,$$

can be computed by constructing a subset $\mathfrak{L} \subseteq \mathcal{L}_\lambda^i(\mu)$ so that

$$P_i(\lambda, \mu) = \sum_{L \in \mathfrak{L}} q^{\deg_i(L)}.$$

Here $\mathcal{L}_\lambda^i(\mu)$ is the set of all non-negative linear combinations of elements in $\Phi^i \cup \Phi^{2i-1} \cup \Phi^{3i-2} \cup \ldots$ (i.e., roots $\alpha \in \Phi^+$ such that $i - 1$ divides $\mathrm{ht}(\alpha) - 1$)) equal to $\lambda - \mu$ and $\deg_i$ is a function defined on any positive root by the formula

$$\deg_i(\alpha) := \frac{\mathrm{ht}(\alpha) - 1}{i-1},$$

and extended $\mathbb{Z}$-linearly to any element in $\mathcal{L}^i_\lambda(\mu)$.

This idea is reminiscent of Deodhar's proposal [3] for a counting formula for Kazhdan-Lusztig polynomials recently refined by the first author and Geordie Williamson [13].

1.4. Structure of the paper

The paper is structured as follows. We start in Section 2.1 by reviewing root systems and (extended) affine Weyl groups (in Example 2.1 we focus on type A and give a more elementary description of these objects). In Section 2.2 we define the spherical Hecke algebra together with its standard and Kazhdan-Lusztig bases and in Section 2.4 we define the pre-canonical bases in detail and prove that they are bases of the spherical Hecke algebra.

In Section 3 we prove Theorem 1.2 (i.e. give a formula for inverse Kazhdan-Lusztig polynomials and for inverse atomic polynomials) using root system combinatorics. In Section 4 we prove Theorem 1.4 by defining several $\mathbf{M}^j_\lambda$ that interpolate between $\mathbf{N}^i_\lambda$ and $\mathbf{N}^{i+1}_\lambda$ and behave particularly well under the Weyl group action. Finally in Section 5 we prove Theorem 1.5 by first defining some bases $\hat{\mathbf{N}}^{i+1}$, then analyzing the decomposition of $\mathbf{N}^i$ in $\mathbf{N}^{i+1}$ and then checking that this decomposition is the same as the one of $\mathbf{N}^i$ in $\hat{\mathbf{N}}^{i+1}$.

1.5. Acknowledgments

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2. Definition of the pre-canonical bases

In this section we introduce the (extended) affine Weyl group and the corresponding Hecke algebras. We refer to [21] and [7] for more details.

2.1. Dominant weights and affine Weyl groups

The reader mostly interested in type A_n can skip this section (that is slightly technical) and read directly Example 2.1 instead. We will need the definitions given in this section to define the spherical Hecke algebra (where the pre-canonical bases live) in Section 2.2. Our definition is equivalent to that given on the literature (cf. Remark 2.3) but it is more natural from the point of view of categorification (cf. Remark 2.4).

Let $(X, \Phi, X^\vee, \Phi^\vee)$ be a reduced root datum where X is the character lattice with roots Φ, and $X^\vee$ is the cocharacter lattice with coroots $\Phi^\vee$. We fix a system of simple roots Δ and positive roots Φ^+. We assume that our root datum is simply connected, i.e. that $X^\vee = \mathbb{Z}\Phi^\vee$. Let ρ be the half-sum of all the positive roots, and $\rho^\vee$ the half-sum of all the positive coroots. Let $\langle -, - \rangle$ denote the pairing between weights X and coweights $X^\vee$.

We denote by $\leq$ the dominance order on X: we say that $\lambda \leq \mu$ if $\mu - \lambda$ can be written as a integral linear combination of elements in Φ^+.

Let $X_\mathbb{R} := X \otimes_\mathbb{Z} \mathbb{R}$. For a root $\alpha \in \Phi$, let s_α denote the corresponding reflection: $s_\alpha : X_\mathbb{R} \to X_\mathbb{R}$ defined as $s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$. Let S_f be the set of reflections s_α, with $\alpha \in \Delta$. The group W_f generated by S_f is the (finite) Weyl group. We denote by w_0 the longest element in W_f.

The affine Weyl group W_a is the subgroup of affine transformations of $X_\mathbb{R}$ generated by W_f and $\mathbb{Z}\Phi$ (acting as translations). We have $W_a \cong W_f \ltimes \mathbb{Z}\Phi$. The group W_a can also be described as the group generated by $s_{\alpha,m}$, the reflections along the hyperplanes

$$H_{\alpha,m} = \{\lambda \in X_\mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = m\},$$

for $\alpha \in \Phi$ and $m \in \mathbb{Z}$. The connected components of the complement of the hyperplanes

$$X_\mathbb{R} \setminus \bigcup_{\alpha,m} H_{\alpha,m}$$

are called alcoves. We call $C_0 = \{\lambda \in X_\mathbb{R} \mid -1 < \langle \lambda, \alpha^\vee \rangle < 0 \text{ for any } \alpha \in \Phi^+\}$ the fundamental alcove. Then the map $w \mapsto wC_0$ defines a bijection between W_a and the set of alcoves.

The walls of C_0 are $H_{\alpha,0}$, for $\alpha \in \Delta$, and $H_{\beta,-1}$ for β the longest short root (so that $\beta^\vee$ is the longest coroot). We set $s_0 := s_{\beta,-1}$ to be the reflection along $H_{\beta,-1}$. Then W_a is a Coxeter group with simple reflections $S = S_f \cup \{s_0\}$.

We also consider the extended affine Weyl group W_e: this is the subgroup of affine transformations of $X_\mathbb{R}$ generated by W_f and X (where X acts as translations). We have $W_e = W_f \ltimes X$.[1] Although W_e is not a Coxeter group in general, one can still define the length $\ell(w)$ of an element $w \in W_e$ by counting how many hyperplanes separate C_0 and wC_0.

Let Ω be the subgroup of length 0 elements in W_e. In other words, this is the subgroup of elements $\sigma \in W_e$ such that $\sigma(C_0) = C_0$. Hence, every element of Ω permutes the walls of C_0, therefore conjugation by Ω permutes the simple reflections in W_a, so Ω can be seen as a group of automorphisms of the Dynkin diagram of W_a.

The group Ω is isomorphic to the fundamental group $X/\mathbb{Z}\Phi$ of the root datum. In fact, for $\lambda \in X$, let $t_\lambda \in W_e$ denote the corresponding translation. Then $w_0(C_0) + \lambda$ is an alcove, so there exists a unique element $\theta(\lambda) \in W_a$ such that $\theta(\lambda)(C_0) = w_0(C_0) + \lambda$.

[1] In the literature (e.g. [1,7]) the extended affine Weyl group is often defined as the group generated by W_f and the coweight lattice $X^\vee$.

The map $\lambda \mapsto \theta(\lambda)^{-1} t_\lambda w_0$ defines a surjective group homomorphism from X to Ω with kernel $\mathbb{Z}\Phi$. It follows that $W_e/W_a \cong X/\mathbb{Z}\Phi \cong \Omega$ and $W_e \cong W_a \rtimes \Omega$.

Moreover, we have $W_e/W_f \cong X$, so $W_f \backslash W_e / W_f \cong X^+$, where X^+ is the set of dominant weights. We can actually refine this bijection. In fact, we have compatible decompositions

$$\begin{array}{ccc} X^+ & \xrightarrow{\sim} & \bigsqcup_{\sigma \in X/\mathbb{Z}\Phi} (\sigma + \mathbb{Z}\Phi) \cap X^+ \\ \downarrow\wr & & \downarrow\wr \\ W_f \backslash W_e / W_f & \xrightarrow{\sim} & \bigsqcup_{\sigma \in \Omega} W_f \backslash W_a / \sigma(W_f) \end{array} \tag{2.1}$$

where $\sigma(W_f) := \sigma W_f \sigma^{-1}$. The bottom horizontal arrow sends the double coset containing t_λ, $\lambda \in X^+$, to the double coset containing $\theta(\lambda)$, for $\sigma = \theta(\lambda)^{-1} t_\lambda w_0$. In particular, for every $\sigma \in \Omega$, the map θ defines a bijection

$$\theta : (\sigma + \mathbb{Z}\Phi) \cap X^+ \xrightarrow{\sim} W_f \backslash W_a / \sigma(W_f).$$

This bijection intertwines the Bruhat order on the right with the dominance order on the left. Moreover, we have

$$\ell(\theta(\lambda)) = \ell(w_0) + 2\langle \lambda, \rho^\vee \rangle$$

(see for example [7, Eq. (2.9)]).

Example 2.1. Since an important part of this paper is devoted exclusively to type A_n, we spell out the definitions in the previous section in this case.

Assume that $n \geq 2$. Let $X_\mathbb{R} \subset \mathbb{R}^{n+1}$ be the subspace of vectors with coordinates adding up to zero. The set of roots is $\Phi := \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n+1\}$. The simple roots are

$$\Delta := \{\alpha_1 := \epsilon_1 - \epsilon_2,\ \alpha_2 := \epsilon_2 - \epsilon_3,\ \ldots\ ,\ \alpha_n := \epsilon_n - \epsilon_{n+1}\},$$

and the fundamental weights are

$$\varpi_i := (\epsilon_1 + \cdots + \epsilon_i) - \frac{i}{n+1} \sum_{j=1}^{n+1} \epsilon_j.$$

The character lattice (or weight lattice) is $X := \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2 + \cdots + \mathbb{Z}\varpi_n$ and the set of dominant weights is $X^+ := \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2 + \cdots + \mathbb{N}\varpi_n$. For $1 \leq i < n+1$, define s_i to be the reflection in $X_\mathbb{R}$ that permutes ϵ_i and ϵ_{i+1} (in other words, the reflection that fixes the hyperplane $x_i + x_{i+1} = 0$.) The Weyl group $W_f \subset \mathrm{End}(X_\mathbb{R})$ is the subgroup of endomorphisms of the vector space $X_\mathbb{R}$ generated by the s_i with $1 \leq i < n+1$. Let s_0 be the reflection through the hyperplane with equation $x_1 + x_{n+1} + 1 = 0$. The affine Weyl group W_a is the subgroup of affine endomorphisms of $X_{\mathbb{R}}$ generated by $S := \{s_i : 0 \leq i \leq n\}$.

The Dynkin diagram of W_a can be seen as a regular $(n+1)$-gon with vertices S. Let $\Omega \cong \mathbb{Z}/(n+1)\mathbb{Z}$ be the group of cyclic permutations of this diagram (see also Remark 2.2). The element $\sigma_i \in \Omega$ for $1 \leq i \leq n+1$ is defined by $\sigma_i(s_j) = s_{i+j}$ (where the sub-index $i+j$ is understood modulo $n+1$).

Let w_0 be the longest element in W_f. We now define a function $\theta : X^+ \to W_a$ as follows. If $\lambda \in X^+ \cap \mathbb{Z}\Phi$, then the translation t_λ is an element of W_a and we define $\theta(\lambda) := t_\lambda w_0$. If $\lambda \in X^+ \setminus \mathbb{Z}\Phi$, then there exists $1 \leq i \leq n$ such that $\lambda = \mu + \varpi_i$ and $\mu \in \mathbb{Z}\Phi \cap X^+$. Then we define $\theta(\lambda) = \theta(\mu) w_0 \theta(\varpi_i)$, where $\theta(\varpi_i)$ is the longest element in $W_f \sigma_i(W_f)$. More explicitly, we have

$$w_0\theta(\varpi_i) = \prod_{k=1}^{i} s_{[1-k, n-i-k+1]},$$

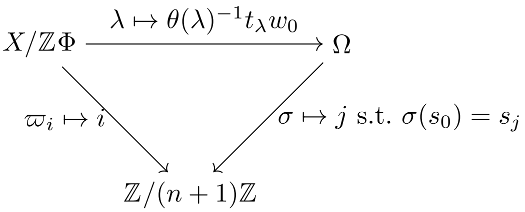
where, for $a \leq b$, we define $s_{[a,b]} := s_a s_{a+1} \cdots s_b$ (again, the sub-indices are understood modulo $n+1$). By construction, we have that the right descent set of $\theta(\lambda)$ is $S \setminus \{s_i\}$.

An important property of this map is that $\lambda \leq \mu$ in the dominance order if and only if $\theta(\lambda) \leq \theta(\mu)$ in the (strong) Bruhat order.

Remark 2.2. If Φ is of type A_n, as mentioned in Example 2.1, the group Ω of length 0 element in W_e is isomorphic to the group $\mathbb{Z}/(n+1)\mathbb{Z}$ which acts on the Dynkin diagram by cycling the simple reflections $\{s_0, s_1, \ldots s_n\}$ of W_a. We explain here how to obtain this isomorphism.

The isomorphism $\Omega \to \mathbb{Z}/(n+1)\mathbb{Z}$ is given by the map $\sigma \mapsto j$, where j is the index of the simple reflection $\sigma(s_0) = \sigma s_0 \sigma^{-1} \in S$. For any $0 \leq j \leq n$, we can define $\sigma_j \in \Omega$, to be the unique element which sends s_0 to s_j. Then, an element $x \in W_a$ is a maximal element in its double coset $W_f x \sigma_j(W_f) \in W_f \backslash W_a / \sigma_j(W_f)$ if and only if its left descent set is $S \setminus \{s_0\}$ and its right descent set is $S \setminus \{s_j\}$.

On the other hand, the fundamental weights ϖ_i, $1 \leq i \leq n$, together with 0 form a set of representatives of $X/\mathbb{Z}\Phi$. So we can canonically identify $X/\mathbb{Z}\Phi$ with $\mathbb{Z}/(n+1)\mathbb{Z}$ by sending ϖ_i to $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ (see [1, Prop VI.2.3.6] for more details). Hence, if we write a weight λ in the basis of fundamental weights as $\lambda = \sum_{i=0}^{n} a_i \varpi_i$, then its class in $\mathbb{Z}/(n+1)\mathbb{Z}$ is given by $\sum_{i=1}^{n} i a_i$. We can summarize this information in the following commutative diagram.



Every fundamental weight ϖ_i is the minimal element in the set $(\sigma+\mathbb{Z}\Phi)\cap X^+$, hence $\theta(\varpi_i)$ is the longest element in the double coset $W_f\sigma_i(W_f)$. More generally, we can compute θ recursively as follows. Assume that we know $\theta(\lambda)$ for $\lambda\in X^+$ and let $\sigma\in\Omega$ be the class of λ, then $\theta(\lambda+\varpi_i)=\theta(\lambda)\sigma(w_0^{-1}\theta(\varpi_i))=\theta(\lambda)\sigma w_0^{-1}\theta(\varpi_i)\sigma^{-1}=t_\lambda\theta(\varpi_1)\sigma^{-1}$. Notice that $\ell(w_0^{-1}\theta(\varpi_i))=2\langle\varpi_i,\rho^\vee\rangle=\ell(\theta(\lambda+\varpi_i))-\theta(\lambda)$, so we obtain a reduced expression of $\theta(\lambda+\varpi_i)$ simply by stacking together a reduced expression of $\theta(\lambda)$ and one of $\sigma(w_0^{-1}\theta(\varpi_i))$. By recursion, in this way one can easily obtain reduced expressions of every $\theta(\lambda)$, for $\lambda\in X^+$.

2.2. The spherical Hecke algebra

Let $\mathcal{H}$ be the Hecke algebra of W_a over $\mathbb{Z}[v,v^{-1}]$ with standard basis $\{\mathbf{H}_x\}$ and Kazhdan-Lusztig basis $\{\underline{\mathbf{H}}_x\}$. Recall that we use the convention $q:=v^2$. The action of Ω on W_a naturally extends to an action by algebra automorphisms on $\mathcal{H}$, where $\sigma\in\Omega$ sends $\mathbf{H}_x$ to $\mathbf{H}_{\sigma(x)}$ for any $x\in W_a$.

Let $\underline{\mathbf{H}}_f:=\underline{\mathbf{H}}_{w_0}$ be the Kazhdan-Lusztig basis element for the longest element $w_0\in W_f$. We define

$$\mathcal{H}^\sigma:=\underline{\mathbf{H}}_f\mathcal{H}\cap\mathcal{H}\sigma(\underline{\mathbf{H}}_f)\subset\mathcal{H}$$

Similarly, for $\sigma,\tau\in\Omega$ we can define ${}^\sigma\mathcal{H}^\tau:=\sigma(\underline{\mathbf{H}}_f)\mathcal{H}\cap\mathcal{H}\tau(\underline{\mathbf{H}}_f)$. Notice that the action by τ induces an isomorphism of $\mathbb{Z}[v,v^{-1}]$-modules $\mathcal{H}^\sigma\cong{}^\tau\mathcal{H}^{\tau\sigma}$.

For a finite subgroup H of W_a we denote by $\pi_H(q)$ the *Poincaré polynomial* of H, defined as $\pi_H(q)=\sum_{w\in H}q^{\ell(w)}$. As in [20, §2.3], we can arrange all the $\mathcal{H}^\sigma$, for $\sigma\in\Omega$, in an Ω-graded algebra

$$\widetilde{\mathcal{H}}:=\bigoplus_{\sigma\in\Omega}\mathcal{H}^\sigma \tag{2.2}$$

where multiplication $\mathcal{H}^\tau\times\mathcal{H}^\sigma\to\mathcal{H}^{\tau\sigma}$ is defined via

$$\begin{gathered}\mathcal{H}^\tau\times\mathcal{H}^\sigma\xrightarrow{\sim}\mathcal{H}^\tau\times{}^\tau\mathcal{H}^{\tau\sigma}\to\mathcal{H}^{\tau\sigma}\\ (x,y)\to(x,\tau(y))\to\frac{v^{\ell(w_0)}}{\pi_{W_f}(v^2)}x\tau(y)\end{gathered}$$

(here $x\tau(y)$ is the product of x and $\tau(y)$ as elements of $\mathcal{H}$). Notice that $\underline{\mathbf{H}}_f$ is the unity of the algebra $\widetilde{\mathcal{H}}$ (cf. [20, Eq. (2.2.4)]). For $\lambda\in X^+$ we define

$$\mathbf{H}_\lambda:=\sum_{x\in W_f\theta(\lambda)\sigma(W_f)}v^{\ell(\theta(\lambda))-\ell(x)}\mathbf{H}_x \tag{2.3}$$

Then $\{\mathbf{H}_\lambda\}$ is the standard basis of $\widetilde{\mathcal{H}}$. For an element $x\in W_a$, we have $\underline{\mathbf{H}}_x\in\mathcal{H}^\sigma$ if and only if x its maximal in its double coset $W_fx\sigma(W_f)$. Since $\theta(\lambda)$ is by definition maximal in its double coset, we can define $\underline{\mathbf{H}}_\lambda := \underline{\mathbf{H}}_{\theta(\lambda)} \in \mathcal{H}^\sigma$. The set $\{\underline{\mathbf{H}}_\lambda\}$ is the Kazhdan-Lusztig basis of $\widetilde{\mathcal{H}}$.

We can write

$$\underline{\mathbf{H}}_\lambda = \sum_{\mu \leq \lambda} h_{\mu,\lambda}(v) \mathbf{H}_\mu$$

where $h_{\lambda,\lambda}(v) = 1$ and $h_{\mu,\lambda}(v) \in v\mathbb{Z}_{\geq 0}[v]$. We set $h_{\mu,\lambda}(v) = 0$ if $\mu \not\leq \lambda$. The polynomials $h_{\mu,\lambda}(v)$ are called Kazhdan-Lusztig polynomials, and coincide with the Kazhdan-Lusztig polynomials of $\mathcal{H}$, i.e. we have $h_{\mu,\lambda}(v) = h_{\theta(\mu),\theta(\lambda)}(v)$.

Remark 2.3. The definition of the spherical Hecke algebra used in [7] is slightly different. Knop defines it as a subring of the *extended affine Hecke algebra*. It is easy to use the bijections in eq. (2.1) to show that the two definitions are equivalent.

Remark 2.4. There is an isomorphism $\widetilde{\mathcal{H}} \cong \mathbb{Z}[v, v^{-1}][X]^{W_f}$, called the Satake isomorphism (see [7,19] for more details). Under this isomorphism, the Kazhdan-Lusztig basis $\underline{\mathbf{H}}_\lambda$ is sent to the Weyl characters and the standard basis $\mathbf{H}_\lambda$ is sent to Hall-Littlewood polynomials of the reductive group G associated with the root datum [12].

The Satake isomorphism admits a categorification, called the *geometric Satake isomorphism* [14], as an equivalence of monoidal categories between the category of finite dimensional representations of G and the category of equivariant perverse sheaves on the affine Grassmannian of the Langlands dual group $G^\vee$.

We can define an additional basis of $\widetilde{\mathcal{H}}$ as follows.

$$\mathbf{N}_\lambda := \sum_{\mu \leq \lambda} v^{2\,\mathrm{ht}(\lambda-\mu)} \mathbf{H}_\mu = \sum_{\mu \leq \lambda} v^{\ell(\theta(\lambda)) - \ell(\theta(\mu))} \mathbf{H}_\mu. \tag{2.4}$$

For $x \in W_a$, define

$$\mathbf{N}_x := \sum_{y \leq x} v^{\ell(x) - \ell(y)} \mathbf{H}_y \in \mathcal{H}. \tag{2.5}$$

It is easy to check that equations eq. (2.4) and eq. (2.5) coincide, i.e. $\mathbf{N}_\lambda = \mathbf{N}_{\theta(\lambda)}$. We can write

$$\underline{\mathbf{H}}_\lambda = \sum_{\mu \leq \lambda} a_{\lambda,\mu}(v) \mathbf{N}_\mu$$

for some polynomials $a_{\lambda,\mu}(v) \in \mathbb{Z}[v]$. The polynomials $a_{\lambda,\mu}(v)$ are called *atomic polynomials* and one says that $\underline{\mathbf{H}}_\lambda$ admits an *atomic decomposition* if $a_{\lambda,\mu}(v) \in \mathbb{Z}_{\geq 0}[v]$ for every $\mu \in X^+$. We remark that we follow the convention from [10], but the sub-indices are inverted in the a polynomial with respect to Kazhdan-Lusztig polynomials, i.e. in the first position one has the biggest weight.

In type $\tilde{A}_n$ there is an atomic decomposition, as proved by Lascoux [9] and Shimozono [18]. In [10] the atomic decomposition in type $\tilde{A}_n$ was reproved using crystals. In their paper, Lecouvey and Lenart also consider other types and, even if the atomic decomposition does in general fail, they conjecture that it still holds under some mild assumptions on λ.

2.3. Kostka-Foulkes polynomials

Kazhdan-Lusztig polynomials in the spherical Hecke algebra can be reinterpreted, up to normalization, as Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$, and in particular they give q-weight multiplicities of the irreducible representations of the reductive group associated to the root datum [12]. To recall the definition of the Kostka-Foulkes polynomials we first need to introduce the (q-analog of) the Kostant partition function.

Definition 2.5. Let $Q^+ \subset X$ be the positive part of the root lattice, i.e. Q^+ is the subset of weights that can be written as $\sum_{i=1}^n c_i\alpha_i$, with $c_i \in \mathbb{Z}_{\geq 0}$. Let $\mathrm{kpf}_q : Q^+ \to \mathbb{Q}[q]$ be the function defined by

$$\prod_{\alpha\in\Phi^+}\left(\sum_{k\geq 0} q^k e^{-k\alpha}\right) = \sum_{\alpha\in Q^+} \mathrm{kpf}_q(\alpha)e^{-\alpha}. \tag{2.6}$$

The function $\mathrm{kpf}_q : Q^+ \to \mathbb{Q}[q]$ is called the *q-analog of the Kostant partition function.* We trivially extend the definition of kpf_q to the set of weights X by imposing that $\mathrm{kpf}_q(\alpha) = 0$ if $\alpha \notin Q^+$.

Definition 2.6. For $\lambda, \mu \in X^+$ with $\mu \leq \lambda$, the *Kostka-Foulkes polynomials* are defined as

$$K_{\lambda,\mu}(q) = \sum_{w\in W_f} (-1)^{\ell(w)}\, \mathrm{kpf}_q(w(\lambda+\rho) - \mu - \rho). \tag{2.7}$$

For any $\lambda, \mu \in X^+$ with $\mu \leq \lambda$ we have $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$. We remark that in general Equation (2.7) does not lead to a polynomial with positive coefficients if $\mu \notin X^+$.

Kostka-Foulkes polynomials are the Kazhdan-Lusztig polynomials in the spherical Hecke algebra. More precisely, we have by [6, Theorem 1.8]

$$h_{\mu,\lambda}(v) = K_{\lambda,\mu}(v^2).$$

(Kato uses the alternative parametrization $h_{\mu,\lambda}(v) = v^{\ell(\theta(\lambda))-\ell(\theta(\mu))} P_{\theta(\mu),\theta(\lambda)}(v^{-2})$ of Kazhdan-Lusztig polynomials.) We remark again that we follow the conventions in the literature and that the Kostka-Foulkes polynomials have (as the atomic polynomials) their sub-indices inverted with respect to the Kazhdan-Lusztig polynomials.

2.4. The pre-canonical bases

We introduce now a new set of bases of the spherical Hecke algebra $\widetilde{\mathcal{H}}$, that we call the pre-canonical bases. Roughly speaking, one can think of these new bases as an interpolation between the standard basis and the Kazhdan-Lusztig basis.

Definition 2.7. We say that a weight $\lambda \in X$ is *singular* if there is an element $s \in W_f$ which fixes $\lambda + \rho$. Equivalently, λ is singular if there is a root α such that $\langle \lambda + \rho, \alpha^\vee \rangle = 0$. A non-singular weight is called *regular*.

Let us recall that the *dot action* (also called the *affine action*) of the finite Weyl group W_f on the set of weights is given by the formula

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Notice that every dominant weight is regular. In the other direction, for $\lambda \in X$ regular we define $w_\lambda \in W_f$ to be the unique element such that $w_\lambda \cdot \lambda$ is dominant. Moreover, let $\overline{\lambda} := w_\lambda \cdot \lambda \in X^+$. We extend the definition of $\underline{\mathbf{H}}_\lambda$ to non-dominant weights.

Definition 2.8. We define

$$\widetilde{\underline{\mathbf{H}}}_\lambda = \begin{cases} (-1)^{\ell(w_\lambda)} \underline{\mathbf{H}}_{\overline{\lambda}}, & \text{if } \lambda \text{ is regular;} \\ 0, & \text{if } \lambda \text{ is singular.} \end{cases}$$

For an integer $i \geq 1$ define $\Phi^{\geq i}$ to be the set of positive roots with height at least i. We are now ready to define the main characters of the present paper.

Definition 2.9 *(Pre-canonical bases).* For $i \geq 2$ and $\lambda \in X^+$ we define

$$\mathbf{N}^i_\lambda := \sum_{I \subset \Phi^{\geq i}} (-v^2)^{|I|} \widetilde{\underline{\mathbf{H}}}_{\lambda - \sum_{\alpha \in I} \alpha}. \tag{2.8}$$

As it will follow from Part 3 of Lemma 2.11, for any $i \geq 2$, the set $\{\mathbf{N}^i_\lambda\}$ is unitriangular with respect to the Kazhdan-Lusztig basis $\{\underline{\mathbf{H}}_\lambda\}$. Therefore, the set $\{\mathbf{N}^i_\lambda\}$ is indeed a basis of the spherical Hecke algebra $\widetilde{\mathcal{H}}$.

Definition 2.10. For $w \in W_f$, let Φ^+_w be the subset of positive roots α such that $w(\alpha) \in \Phi^+$ and let $\Phi^+_{-w} = \Phi^+ \setminus \Phi^+_w$.

For a subset $I \subset \Phi$ let $\Sigma_I := \sum_{\alpha \in I} \alpha$.

Lemma 2.11. *Let $\lambda \in X^+$ and $I \subset \Phi^+$. Then*

1. $\overline{\lambda - \Sigma_I} \leq \lambda$.

2. $\overline{\lambda - \Sigma_I} = \lambda$ *if and only if there exists* $w \in W_f$ *such that* $I = \Phi^+_{-w}$ *and* $w(\lambda) = \lambda$.
3. For $i \geq 2$, *if* $I \subset \Phi^{\geq i}$ *and* $I \neq \emptyset$, *then* $\overline{\lambda - \Sigma_I} < \lambda$.

Proof. This is proved in [17, Theorem 2.1]. We rewrite here the proof for convenience.

We have $\overline{\lambda - \Sigma_I} = w(\lambda+\rho-\Sigma_I)-\rho$ for $w = w_{\lambda-\Sigma_I}$. Since $w(\lambda) \leq \lambda$ and $w(\rho-\Sigma_I) \leq \rho$ by [8, Lemma 5.9], we have $\overline{\lambda - \Sigma_I} \leq \lambda$. Moreover, $w(\rho - \Sigma_I) = \rho$ if and only if $I = \Phi^+_{-w}$ by [17, Lemma 4.8]. Hence $\overline{\lambda - \Sigma_I} = \lambda$ if and only if there exists an element $w \in W_f$ such that $I = \Phi^+_{-w}$ and $w(\lambda) = \lambda$. This shows the first two parts of the Lemma.

For the last part, notice that for $w \neq id$, the set Φ^+_{-w} always contains a simple root, so it cannot be contained in $\Phi^{\geq i}$ for $i \geq 2$. □

We would like to define $\mathbf{N}^1_\lambda$ similarly to eq. (2.8), but we need to take into account that Part 3 of Lemma 2.11 does not hold for $i = 1$ since there exists subsets $I \subset \Phi^{\geq 1}$ with $I = \Phi^+_{-w}$ and $w \in W^f$. This leads to the following definition.

Definition 2.12 *(First pre-canonical basis).* For any $\lambda \in X^+$ we define

$$\mathbf{N}^1_\lambda := \frac{1}{\pi_{W^\lambda}(v^2)} \sum_{I \subset \Phi^+} (-v^2)^{|I|} \underline{\widetilde{\mathbf{H}}}_{\lambda - \sum_{\alpha \in I} \alpha},$$

where $\pi_{W^\lambda}(q)$ is the Poincaré polynomial of $W^\lambda := \mathrm{stab}_{W_f}(\lambda)$.

Notice that $|\Phi^+_{-w}| = \ell(w)$. By Lemma 2.11(2) we see that the coefficient of $\underline{\mathbf{H}}_\lambda$ in $\mathbf{N}^1_\lambda$ is

$$\frac{1}{\pi_{W^\lambda}(v^2)} \sum_{w \in W^\lambda} (-v^2)^{|\Phi^+_{-w}|}(-1)^{\ell(w)} = 1.$$

Hence, by unitriangularity, the set $\{\mathbf{N}^1_\lambda\}$ is also a basis of $\widetilde{\mathcal{H}}$.

Remark 2.13. It is easy to see that our definition of $\mathbf{N}^1_\lambda$ coincides, after applying the Satake isomorphism, with the definition of the Hall-Littlewood polynomials (compare e.g. with [19, Eq. (1)]). This immediately implies that $\mathbf{N}^1_\lambda$ is the standard basis element $\mathbf{H}_\lambda$, proving the first half of Theorem 1.2. For the reader's convenience we reprove this fact in Theorem 3.5 avoiding the recourse to the Satake isomorphism.

Definition 2.14. We call $\{\mathbf{N}^i_\lambda : \lambda \in X^+\}$ the i^{th} *pre-canonical basis.*

3. The first and second pre-canonical bases

From the definition of the $\mathbf{N}^i_\lambda$ it is clear that if $i > m$, where m is the height of the highest root in Φ^+, we have $\mathbf{N}^i_\lambda = \underline{\mathbf{H}}_\lambda$. The goal of this section is to show that also $\mathbf{N}^1_\lambda$ and $\mathbf{N}^2_\lambda$ are in fact familiar (and previously introduced) objects. Namely, $\mathbf{N}^1_\lambda$

and $\mathbf{N}^2_\lambda$ are respectively the standard basis and the $\mathbf{N}$-basis eq. (2.4) of $\widetilde{\mathcal{H}}$. We start by considering $\mathbf{N}^2$.

Theorem 3.1. *(Anti-atomic formula) For every $\lambda \in X^+$ we have*

$$\mathbf{N}_\lambda = \mathbf{N}^2_\lambda = \sum_{I \subset \Phi^{\geq 2}} (-v^2)^{|I|} \underline{\widetilde{\mathbf{H}}}_{\lambda - \Sigma_I}. \tag{3.1}$$

Recall Definition 2.10. Define $\Delta_w := \Delta \cap \Phi^+_w$ and $\Delta_{-w} := \Delta \cap \Phi^+_{-w}$. Following [5, §24.1], we define $\mathfrak{X}$ be the space of $\mathbb{Q}[q]$-valued functions f on X whose support (defined to be the set of $x \in X$ for which $f(x) \neq 0$) is included in a finite union of sets of the form $\{\lambda - \sum_{\alpha \in \Phi^+} k_\alpha \alpha,\ k_\alpha \in \mathbb{Z}_{\geq 0}\}$. (Such a set is the set of weights occurring in a Verma module $Z(\lambda)$). In other words, an element of $\mathfrak{X}$ can be written as $\sum_{\mu \in X} c_\mu(q) e^\mu$, where $c_\mu(q) \in \mathbb{Q}[q]$ and such that the set of μ for which $c_\mu \neq 0$ is contained in a finite union of sets of the form $\{\lambda - \sum_{\alpha \in \Phi^+} k_\alpha \alpha, k_\alpha \in \mathbb{Z}_{\geq 0}\}$. Then $\mathfrak{X}$ is a commutative $\mathbb{Q}[q]$ algebra. If $\alpha \in \Phi^+$, the element $(1 - qe^{-\alpha})$ is invertible in $\mathfrak{X}$ and we have

$$(1 - qe^{-\alpha})^{-1} = \sum_{k \geq 0} q^k e^{-k\alpha}. \tag{3.2}$$

Similarly, $(q - e^\alpha) \in \mathfrak{X}$ is also invertible and we have

$$(q - e^\alpha)^{-1} = -e^{-\alpha} \sum_{k \geq 0} q^k e^{-k\alpha}. \tag{3.3}$$

For any $\lambda \in X^+$, we consider the following element in $\mathfrak{X}$.

$$\Theta(\lambda) = \sum_{w \in W_f} \frac{e^{w(\lambda)} \prod_{\alpha \in \Phi^{\geq 2}} (1 - qe^{-w(\alpha)})}{\prod_{\alpha \in \Phi^+_w} (1 - qe^{-w(\alpha)}) \prod_{\alpha \in \Phi^+_{-w}} (q - e^{-w(\alpha)})}$$

For $f \in \mathfrak{X}$, we write $f|_{X^+}$ for its restriction to X^+ (i.e. if $f = \sum_{\mu \in X} c_\mu e^\mu$, then $f|_{X^+} = \sum_{\mu \in X^+} c_\mu e^\mu$). We will expand $\Theta(\lambda)|_{X^+}$ in two different ways, and this will lead to our theorem. The first expansion is given by eq. (3.10).

$$\Theta(\lambda)|_{X^+} = \sum_{\substack{I \subset \Phi^{\geq 2} \\ \lambda - \Sigma_I \text{ regular}}} (-q)^{|I|} (-1)^{\ell(w_{\lambda - \Sigma_I})} \sum_{\mu \in X^+} h_{\mu, \overline{\lambda - \Sigma_I}}(q^{\frac{1}{2}}) e^\mu.$$

The second expansion is given by eq. (3.14).

$$\Theta(\lambda)|_{X^+} = \sum_{\substack{\mu \in X^+ \\ \mu \leq \lambda}} q^{\mathrm{ht}(\lambda - \mu)} e^\mu.$$

For $\mu \in X^+$ such that $\mu \leq \lambda$, by comparing the coefficient of e^μ in eq. (3.10) and eq. (3.14) we obtain:

$$q^{\mathrm{ht}(\lambda-\mu)} = \sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I \text{ regular}}} (-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})} h_{\mu,\overline{\lambda-\Sigma_I}}(q^{\frac{1}{2}}). \tag{3.4}$$

Remark 3.2. The rational function $\Theta(\lambda)$ is a q-deformation of the *layer sum polynomials*

$$\Theta(\lambda)_{q=1} = \sum_{w\in W_f} \frac{e^{w(\lambda)}}{\prod_{\alpha\in\Delta}(1-e^{-w(\alpha)})}.$$

We also have $\Theta(\lambda)_{q=1} = \sum e^{\mu}$, where the sum runs over all weights $\mu \in X$ lying in the convex hull of the orbit $W_f \cdot \lambda$. This was first obtained by Postnikov [15, Theorem 4.3] using Brion's formula for counting lattice points in rational polytopes [2] and later reproved by Schützer [17] using root system combinatorics.

Our strategy for proving Theorem 3.1 is based on Schützer's approach. We have carefully chosen the q-deformation in $\Theta(\lambda)$ so that, when restricted to dominant weights, it gives the desired q-deformation of the RHS of [17, Eq. (10)] (cf. Equation (3.10)).

Proof of Theorem 3.1. On the right side of eq. (3.1) we have

$$\begin{aligned}\sum_{I\subset\Phi^{\geq 2}} (-v^2)^{|I|}\underline{\widetilde{\mathbf{H}}}_{\lambda-\Sigma_I} &= \sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I \text{ regular}}} (-v^2)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})}\underline{\mathbf{H}}_{\overline{\lambda-\Sigma_I}} \\ &= \sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I \text{ regular}}} (-v^2)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})} \sum_{\substack{\mu\leq\overline{\lambda-\Sigma_I}\\ \mu\in X^+}} h_{\mu,\overline{\lambda-\Sigma_I}}(v)\mathbf{H}_{\mu}.\end{aligned}$$

Hence, for every $\mu \in X^+$ such that $\mu \leq \lambda$, the coefficient of $\mathbf{H}_\mu$ in the right side is

$$\sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I \text{ regular}}} (-v^2)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})} h_{\mu,\overline{\lambda-\Sigma_I}}(v). \tag{3.5}$$

Applying eq. (3.4) for $q = v^2$ we see that eq. (3.5) is the same as $v^{2\,\mathrm{ht}(\lambda-\mu)}$, the coefficient of $\mathbf{H}_\mu$ in $\mathbf{N}_\lambda$. The identity in Theorem 3.1 follows. □

3.1. The first expansion of $\Theta(\lambda)$

We have

$$\prod_{\alpha\in\Phi^{\geq 2}} (1-qe^{-w(\alpha)}) = \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|} e^{-w(\Sigma_I)}. \tag{3.6}$$

Using eq. (3.6), eq. (3.2) and eq. (3.3) we can rewrite $\Theta(\lambda)$ as

$$\Theta(\lambda) = \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|} \sum_{w\in W_f} e^{w(\lambda-\Sigma_I)} \prod_{\alpha\in\Phi_w^+} \left(\sum_{k\geq 0} q^k e^{-kw(\alpha)}\right) \times \prod_{\alpha\in\Phi_{-w}^+} \left(-e^{w(\alpha)} \sum_{k\geq 0} q^k e^{kw(\alpha)}\right) \tag{3.7}$$

Recall that for every $w \in W_f$ we have $|\Phi_{-w}^+| = \ell(w)$ and $w(\rho) - \rho = \sum_{\alpha\in\Phi_{-w}^+} w(\alpha)$. Hence, we have

$$\prod_{\alpha\in\Phi_{-w}^+} -e^{w(\alpha)} = (-1)^{\ell(w)} e^{w(\rho)-\rho}. \tag{3.8}$$

We can rewrite $\Theta(\lambda)$ once again as follows

$$\begin{aligned}\Theta(\lambda) &= \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|} \sum_{w\in W_f} (-1)^{\ell(w)} e^{w(\lambda-\Sigma_I+\rho)-\rho} \prod_{\alpha\in\Phi_w^+} \left(\sum_{k\geq 0} q^k e^{-kw(\alpha)}\right) \\ &\quad \times \prod_{\alpha\in\Phi_{-w}^+} \left(\sum_{k\geq 0} q^k e^{kw(\alpha)}\right) \\ &= \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|} \sum_{w\in W_f} (-1)^{\ell(w)} e^{w(\lambda-\Sigma_I+\rho)-\rho} \prod_{\beta\in\Phi^+} \left(\sum_{k\geq 0} q^k e^{-k\beta}\right) \\ &= \sum_{I\subset\Phi^{\geq 2}} (-q)^{|I|} \sum_{w\in W_f} (-1)^{\ell(w)} e^{w(\lambda-\Sigma_I+\rho)-\rho} \sum_{\beta\in Q^+} \mathrm{kpf}_q(\beta) e^{-\beta}.\end{aligned}$$

The first equality follows from eq. (3.8) applied to the last term of the right hand side of eq. (3.7). For the second equality replace $\beta = w(\alpha)$ and use the decomposition $\Phi^+ = w(\Phi_w^+) \dot\cup (-w(\Phi_{-w}^+))$. The last equation follows directly from equation (2.6). The next Lemma implies that in previous sum only the subsets I such that $\lambda - \Sigma_I$ is regular need to be considered.

Lemma 3.3. *Assume $\nu \in X$ is singular, then*

$$\sum_{w\in W_f} (-1)^{\ell(w)} e^{w(\nu+\rho)} = 0$$

Proof. Let $W^{\nu+\rho}$ be the stabilizer of $\nu+\rho$ in W_f. We denote by $W_{\nu+\rho}$ the set of minimal length representatives in $W_f/W^{\nu+\rho}$. Multiplication induces a length preserving bijection $W_{\nu+\rho} \times W^{\nu+\rho} \cong W_f$. We have

$$\sum_{w\in W_f} (-1)^{\ell(w)} e^{w(\nu+\rho)} = \sum_{x\in W_{\nu+\rho}} (-1)^{\ell(x)} e^{x(\nu+\rho)} \sum_{y\in W^{\nu+\rho}} (-1)^{\ell(y)}$$

It is enough to show that $\sum_{y\in W^{\nu+\rho}}(-1)^{\ell(y)}=0$. Notice that

$$\sum_{y\in W^{\nu+\rho}}(-1)^{\ell(y)}=|\{y\in W^{\nu+\rho}\mid \ell(y)\text{ even}\}|-|\{y\in W^{\nu+\rho}\mid \ell(y)\text{ odd}\}|.$$

The group $W^{\nu+\rho}$ is a reflection subgroup (it is generated by the reflections in W_f fixing $\nu+\rho$). If ν is singular then $W^{\nu+\rho}$ is non-trivial and contains a reflection s. Multiplication by s induces a bijection between elements of even length and elements of odd length in $W^{\nu+\rho}$. $\square$

If ν is now an arbitrary regular weight, we have

$$\sum_{w\in W_f}(-1)^{\ell(w)}e^{w(\nu+\rho)-\rho}=(-1)^{\ell(w_\nu)}\sum_{w\in W_f}(-1)^{\ell(w)}e^{w(\overline{\nu}+\rho)-\rho}$$

So, if $\lambda-\Sigma_I$ is regular, we have

$$\begin{aligned}
&\sum_{w\in W_f}(-1)^{\ell(w)}e^{w(\lambda-\Sigma_I+\rho)-\rho}\sum_{\beta\in Q^+}\mathrm{kpf}_q(\beta)e^{-\beta}=\\
&\quad=(-1)^{\ell(w_{\lambda-\Sigma_I})}\sum_{w\in W_f}(-1)^{\ell(w)}e^{w(\overline{\lambda-\Sigma_I}+\rho)-\rho}\sum_{\beta\in Q^+}\mathrm{kpf}_q(\beta)e^{-\beta}\\
&\quad=(-1)^{\ell(w_{\lambda-\Sigma_I})}\sum_{\substack{\mu\in X\\ \mu\leq\overline{\lambda-\Sigma_I}}}\left(\sum_{w\in W_f}(-1)^{\ell(w)}\,\mathrm{kpf}_q(w(\overline{\lambda-\Sigma_I}+\rho)-\rho-\mu)\right)e^{\mu}. \quad (3.9)
\end{aligned}$$

The last equation is obtained by recalling that $\mathrm{kpf}_q(\alpha)$ is defined to be zero if $\alpha\in X\setminus Q^+$ and by noticing that for any $w\in W_f$ we have

$$w\cdot\overline{\lambda-\Sigma_I}\leq\overline{\lambda-\Sigma_I}.$$

Finally, we restrict to $\Theta(\lambda)|_{X^+}$. Applying the definition of the Kostka-Foulkes polynomials eq. (2.7) we obtain

$$\begin{aligned}
\Theta(\lambda)|_{X^+}&=\sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I\text{ regular}}}(-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})}\sum_{\mu\in X^+}K_{\overline{\lambda-\Sigma_I},\mu}(q)e^{\mu}\\
&=\sum_{\substack{I\subset\Phi^{\geq 2}\\ \lambda-\Sigma_I\text{ regular}}}(-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})}\sum_{\mu\in X^+}h_{\mu,\overline{\lambda-\Sigma_I}}(q^{\frac{1}{2}})e^{\mu}. \quad (3.10)
\end{aligned}$$

3.2. The second expansion of $\Theta(\lambda)$

Let $\Phi^{\geq 2}_{-w} := \Phi^+_{-w} \setminus \Delta_{-w}$. We start by rewriting $\Theta(\lambda)$ as

$$\Theta(\lambda) = \sum_{w\in W_f} \frac{e^{w(\lambda)} \prod_{\alpha\in\Phi^+_w\setminus\Delta_w}(1-qe^{-w(\alpha)}) \prod_{\alpha\in\Phi^+_{-w}\setminus\Delta_{-w}}(1-qe^{-w(\alpha)})}{\prod_{\alpha\in\Phi^+_w}(1-qe^{-w(\alpha)}) \prod_{\alpha\in\Phi^+_{-w}}(q-e^{-w(\alpha)})}$$

$$= \sum_{w\in W_f} \frac{e^{w(\lambda)}}{\prod_{\alpha\in\Delta_w}(1-qe^{-w(\alpha)}) \prod_{\alpha\in\Delta_{-w}}(q-e^{-w(\alpha)})} \prod_{\alpha\in\Phi^{\geq 2}_{-w}} \frac{1-qe^{-w(\alpha)}}{q-e^{-w(\alpha)}}$$

For every $\alpha \in \Phi^+_{-w}$ we have

$$\frac{1-qe^{-w(\alpha)}}{q-e^{-w(\alpha)}} = \sum_{k\geq 0} p_k(q) e^{kw(\alpha)}$$

where $p_0(q) = q$ and $p_k(q) = q^{k+1} - q^{k-1}$ if $k \geq 1$. Now we can use eq. (3.2) and eq. (3.3) to rewrite:

$$\Theta(\lambda) = \sum_{w\in W_f} e^{w(\lambda)} \prod_{\alpha\in\Delta_w} \left(\sum_{k\geq 0} q^k e^{-kw(\alpha)}\right) \prod_{\alpha\in\Delta_{-w}} \left(-e^{w(\alpha)} \sum_{k\geq 0} q^k e^{kw(\alpha)}\right)$$
$$\times \prod_{\alpha\in\Phi^{\geq 2}_{-w}} \left(\sum_{k\geq 0} p_k(q) e^{kw(\alpha)}\right) \tag{3.11}$$

The terms e^μ which occur in the sum for a fixed $w \in W_f$ are for μ of the following form

$$\mu = w(\lambda) - \sum_{\alpha\in\Delta_w} k_\alpha w(\alpha) + \sum_{\beta\in\Delta_{-w}} (k_\beta+1) w(\beta) + \sum_{\gamma\in\Phi^{\geq 2}_{-w}} k_\gamma w(\gamma),$$

with $k_\alpha, k_\beta, k_\gamma \geq 0$.

Let P^λ be the set of weights of an irreducible representation of highest weight λ of the reductive group G associated to the root datum. In other words, P^λ is the set of weights $\mu \in X$ such that $w(\mu) \leq \lambda$ for every $w \in W_f$.

The following Lemma is an adaptation of [17, Lemma 4.2].

Lemma 3.4. *Let $\lambda \in X^+$. Assume that there is an element $\mu \in P^\lambda$ such that*

$$\mu = w(\lambda) - \sum_{\alpha\in\Delta_w} k_\alpha w(\alpha) + \sum_{\beta\in\Delta_{-w}} (k_\beta+1) w(\beta) + \sum_{\gamma\in\Phi^{\geq 2}_{-w}} k_\gamma w(\gamma) \tag{3.12}$$

for some $k_\alpha, k_\beta, k_\gamma \geq 0$ and $w \in W_f$. Then $w = id$.

Proof. Since $\mu \in P^\lambda$, we have $\lambda \geq w^{-1}(\mu)$, so $\lambda - w^{-1}(\mu) = \sum_{\alpha\in\Delta} n_\alpha \alpha$, for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. Applying w we obtain

$$w(\lambda) - \mu = \sum_{\alpha\in\Delta} n_\alpha w(\alpha). \tag{3.13}$$

Now assume by contradiction that $w \neq id$. Then $\Delta_{-w} \neq \emptyset$ and we choose $\beta \in \Delta_{-w}$. When we write $w(\lambda) - \mu$ in the basis $\{w(\alpha)\}_{\alpha\in\Delta}$ of $X_\mathbb{R}$, by eq. (3.13) the coefficient of $w(\beta)$ is $n_\beta \geq 0$.

On the other hand, using eq. (3.12) we see that the coefficient of $w(\beta)$ in $w(\lambda) - \mu$ is

$$-k_\beta - 1 - \sum_{\gamma\in\Phi^{\geq 2}_{-w}} k_\gamma [\gamma]_\beta < 0,$$

where $[\gamma]_\beta$ denotes the coefficient of β when γ is written in the basis of simple roots (clearly $[\gamma]_\beta \geq 0$). We get a contradiction, hence $w = id$. □

We restrict ourselves to consider $\Theta(\lambda)|_{X^+}$. Thanks to eq. (3.9) we know that all the weights e^μ occurring in $\Theta(\lambda)|_{X^+}$ satisfy $\mu \leq \overline{\lambda - \Sigma_I}$ for some $I \subset \Phi^{\geq 2}$ and therefore, by Lemma 2.11, they satisfy $\mu \leq \lambda$. Since $\mu \in X^+$ we also have $\mu \in P^\lambda$.

Finally, from Lemma 3.4 it follows that only the term for $w = id$ contributes in eq. (3.11) to the coefficient of e^μ in $\Theta(\lambda)$, for $\mu \in P^\lambda$. Notice that for $w = id$ we have $\Delta_{-w} = \Phi^{\geq 2}_{-w} = \emptyset$. It follows that

$$\Theta(\lambda)|_{X^+} = \left(e^\lambda \prod_{\alpha\in\Delta} \left(\sum_{k\geq 0} q^k e^{-k\alpha} \right)\right)\Bigg|_{X^+} = \sum_{\substack{\mu\in X^+ \\ \mu\leq\lambda}} q^{\mathrm{ht}(\lambda-\mu)} e^\mu. \tag{3.14}$$

3.3. The first pre-canonical basis

We can employ similar techniques to those of Section 3 to show that the first pre-canonical basis coincides with the standard basis, as mentioned in the first part of Theorem 1.2. As pointed out in Remark 2.13, this can be deduced via the Satake transform, using the definition of the Hall-Littlewood polynomials. Here we give a proof which does not pass through Satake.

Theorem 3.5. *For every $\lambda \in X^+$ we have $\mathbf{N}^1_\lambda = \mathbf{H}_\lambda$.*

We start by considering the following element of $\mathfrak{X}$.

$$\Theta_1(\lambda) = \sum_{w\in W_f} \frac{e^{w(\lambda)} \prod_{\alpha\in\Phi^+}(1 - qe^{-w(\alpha)})}{\prod_{\alpha\in\Phi^+_w}(1 - qe^{-w(\alpha)}) \prod_{\alpha\in\Phi^+_{-w}}(q - e^{-w(\alpha)})}.$$

We have

$$\prod_{\alpha\in\Phi^+}(1-qe^{-w(\alpha)}) = \sum_{I\subset\Phi^+}(-q)^{|I|}e^{-w(\Sigma_I)}$$

The functions $\Theta_1(\lambda)$ and $\Theta(\lambda)$ share the same denominator, so working as in Section 3.1 we obtain

$$\Theta_1(\lambda)|_{X^+} = \sum_{\substack{I\subset\Phi^+ \\ \lambda-\Sigma_I \text{ regular}}}(-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})}\sum_{\mu\in X^+}h_{\mu,\overline{\lambda-\Sigma_I}}(q^{\frac{1}{2}})e^{\mu}. \tag{3.15}$$

On the other hand, we have

$$\Theta_1(\lambda) = \sum_{w\in W_f}e^{w(\lambda)}\prod_{\alpha\in\Phi^+_{-w}}\frac{1-qe^{-w(\alpha)}}{q-e^{-w(\alpha)}} = \sum_{w\in W_f}e^{w(\lambda)}\prod_{\alpha\in\Phi^+_{-w}}\left(\sum_{k\geq 0}p_k(q)e^{kw(\alpha)}\right).$$

For a fixed $w\in W_f$, the terms e^{μ} which occur in the sum are for μ of the form

$$\mu = w(\lambda) + \sum_{\alpha\in\Phi^+_{-w}}k_\alpha w(\alpha) \quad \text{with } k_\alpha\in\mathbb{Z}_{>0}.$$

We need a slight modification of Lemma 3.4.

Lemma 3.6. *Let $\lambda\in X^+$. Assume there exists $\mu\in P^\lambda\cap X^+$ such that*

$$\mu = w(\lambda) + \sum_{\alpha\in\Phi^+_{-w}}k_\alpha w(\alpha)$$

for some $k_\alpha\geq 0$ and $w\in W_f$. Then $\mu = w(\lambda) = \lambda$ and $k_\alpha = 0$ for all $\alpha\in\Phi^+_{-w}$.

Proof. Let μ be as in the statement. If $w = id$ then $\Phi^+_{-w} = \emptyset$, and the lemma follows. Assume now that $w\neq id$ and that $k_\gamma > 0$ for some $\gamma\in\Phi^+_{-w}$. Then there exists $\beta\in\Delta_{-w}$ such that the coefficient $[\gamma]_\beta$ is strictly positive (otherwise $\gamma\in\sum_{\alpha\in\Delta_w}\mathbb{Z}_{\geq 0}\alpha$ and $w(\gamma)\in\Phi^+$). Now, as in the proof of Lemma 3.4, by looking at the coefficient of $w(\beta)$ in $w(\lambda)-\mu$ (again, in the basis $\{w(\alpha)\}_{\alpha\in\Delta}$ of $X_\mathbb{R}$), we deduce that it must be non-negative and negative, so we obtain a contradiction. It follows that $k_\alpha = 0$ for all $\alpha\in\Phi^+_{-w}$. Hence, we have $\mu = w(\lambda)\in X^+$. But λ is the only dominant weight in its W_f-orbit, so $w(\lambda) = \lambda$. □

Let W^λ be the stabilizer of λ in W_f. We want to compute the coefficient of e^μ in $\Theta_1(\lambda)$ for $\mu\in X^+$. By eq. (3.15) and Lemma 2.11 we know that it is non-zero only if $\mu\leq\lambda$. Notice that $P^\lambda\cap X^+ = \{\mu\in X^+ \mid \mu\leq\lambda\}$, so, by Lemma 3.6 we deduce that the coefficient of e^μ in $\Theta_1(\lambda)$ is non-zero only if $\mu = \lambda$. Recall that $p_0(q) = q$. We conclude that

$$\Theta_1(\lambda)|_{X^+} = \sum_{w\in W^\lambda} q^{\ell(w)} e^\lambda = \pi_{W^\lambda}(q) e^\lambda. \tag{3.16}$$

Proof of Theorem 3.5. Comparing the coefficient of e^μ in eq. (3.15) and eq. (3.16) we see that for any $\mu \in X^+$ we have

$$\delta_{\mu,\lambda}\pi_{W^\lambda}(q) = \sum_{I\subset\Phi^+} (-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})} \sum_{\mu\in X^+} h_{\mu,\overline{\lambda-\Sigma_I}}(q^{\frac{1}{2}}). \tag{3.17}$$

Then we conclude in a similar vein as in the proof of Theorem 3.1 (just after eq. (3.4)), i.e. the left hand side of eq. (3.17) (divided by $\pi_{W^\lambda}(q)$) gives the coefficient of $\mathbf{H}_\mu$ in $\mathbf{H}_\lambda$ and the right hand side (divided by $\pi_{W^\lambda}(q)$) gives the coefficient of $\mathbf{H}_\mu$ in $\mathbf{N}^1_\lambda$. □

The following corollary is the second part of Theorem 1.2 and it is an easy consequence of Theorem 3.1 and Theorem 3.5.

Corollary 3.7. *For $\lambda \in X^+$ we have the equation*

$$\mathbf{N}^2_\lambda = \sum_{\substack{\mu\in X^+\\ \mu\leq\lambda}} q^{\mathrm{ht}(\lambda-\mu)}\mathbf{N}^1_\mu.$$

Remark 3.8. It seems natural to define for every $i \geq 1$ the rational function

$$\Theta_i(\lambda) := \frac{e^{w(\lambda)}\prod_{\alpha\in\Phi^{\geq i}}(1-qe^{-w(\alpha)})}{\prod_{\alpha\in\Phi^+_w}(1-qe^{-w(\alpha)})\prod_{\alpha\in\Phi^+_{-w}}(q-e^{-w(\alpha)})},$$

so that $\Theta(\lambda) = \Theta_2(\lambda)$ However, although the first expansion easily generalizes to every $i \geq 1$, and we have

$$\Theta_i(\lambda)|_{X^+} = \sum_{\substack{I\subset\Phi^{\geq i}\\ \lambda-\Sigma_I \text{ regular}}} (-q)^{|I|}(-1)^{\ell(w_{\lambda-\Sigma_I})} \sum_{\mu\in X^+} h_{\mu,\overline{\lambda-\Sigma_I}}(q^{\frac{1}{2}})e^\mu,$$

we do not have any interpretation for the second expansion when $i > 2$.

4. Upper half decompositions in type A

In this section we provide a closed formula for the decompositions of $\mathbf{N}^{i+1}_\lambda$ in terms of $\mathbf{N}^i_\lambda$ for all $n/2+1 \leq i \leq n$ in type A_n, thus proving Theorem 1.4. For the rest of this section we fix n and i.

Definition 4.1. Let us define, for $A \subset \Phi$ and $\mu \in X$ the element

$$\mathbf{M}_\mu^A := \sum_{I \subset A} (-q)^{|I|} \underline{\widetilde{\mathbf{H}}}_{\mu - \Sigma_I} \in \mathcal{H}.$$

Lemma 4.2. *For $\lambda \in X$ and $s \in S_f$ a simple reflection of W_f we have the equation*

$$\underline{\widetilde{\mathbf{H}}}_\lambda = -\underline{\widetilde{\mathbf{H}}}_{s\cdot\lambda}.$$

Proof. If λ is singular, then $s \cdot \lambda$ is also singular. Then, by definition, both $\underline{\widetilde{\mathbf{H}}}_\lambda$ and $\underline{\widetilde{\mathbf{H}}}_{s\cdot\lambda}$ vanish and the lemma follows.
Assume now that λ is regular. Then $s \cdot \lambda$ is also regular. We have $w_{s\cdot\lambda} = w_\lambda s$ since $(w_\lambda s) \cdot (s \cdot \lambda) = w_\lambda \cdot \lambda \in X^+$. Hence, $\ell(w_{s\cdot\lambda}) = \ell(w_\lambda) \pm 1$ and we conclude by using the definition of $\underline{\widetilde{\mathbf{H}}}_\lambda$. □

We introduce two notations.
Let $\mu_1, \mu_2, \ldots, \mu_n$ be the coordinates of $\mu \in X$ when expressed in the basis of fundamental weights, i.e. we have $\mu = \sum \mu_j \varpi_j$. It is not hard to prove that $s_j \cdot \mu = \mu$ if and only if $\mu_j = -1$.
For $1 \leq j \leq k \leq n$ define $\alpha_{j,k} := \alpha_j + \alpha_{j+1} + \cdots + \alpha_k$ (all positive roots are of this form in type A_n). Notice that when written in the basis of fundamental weights, $\alpha_{j,k}$ has a 1 in positions j and k, a -1 in positions $j-1$ (if $1 \leq j-1$) and $k+1$ (if $k+1 \leq n$) and 0 elsewhere.

Proposition 4.3. *Let $A \subset \Phi$ and $\mu \in X$. We have the equality*

$$\mathbf{M}_\mu^A = -\mathbf{M}_{s_k\cdot\mu}^{s_k(A)},$$

for all $1 \leq k \leq n$. In particular, if $A = s_k(A)$ and $\mu_k = -1$, we have $\mathbf{M}_\mu^A = 0$.

Proof. The proposition is a direct consequence of Lemma 4.2 and the definition of the element $\mathbf{M}_\mu^A$. □

For $1 \leq j \leq n-i+1$ we set $\gamma_j := \alpha_{j,j+i-1}$. We define $\Gamma_j := \Phi^{>i} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_j\}$ and $\Gamma_0 := \Phi^{>i}$. To shorten notation we define

$$\mathbf{M}_\lambda^j := \mathbf{M}_\lambda^{\Gamma_j}.$$

Lemma 4.4. *Let $\lambda \in X^+$. For $j \geq 1$, if $\lambda_{j+i-1} = 0$ then*

$$\mathbf{M}_\lambda^j = \mathbf{M}_\lambda^{j-1}. \tag{4.1}$$

Proof. The disjoint union $\Gamma_j = \Gamma_{j-1} \cup \{\gamma_j\}$ give us

$$\mathbf{M}_\lambda^j = \mathbf{M}_\lambda^{j-1} - q\mathbf{M}_{\lambda-\gamma_j}^{j-1}. \tag{4.2}$$

On the other hand, we have $s_{j+i-1}(\Gamma_{j-1}) = \Gamma_{j-1}$ for all $1 \leq j \leq n-i+1$. As by hypothesis $\lambda_{j+i-1} = 0$, we can use Proposition 4.3 to conclude that $\mathbf{M}^{j-1}_{\lambda-\gamma_j} = 0$. Therefore, eq. (4.2) reduces to eq. (4.1) and the Lemma is proved. □

Lemma 4.5. *Let $\lambda \in X^+$. For $j \geq 1$, if $\lambda_{j+i-1} > 0$ then*

$$\mathbf{M}^{j}_{\lambda} = \mathbf{M}^{j-1}_{\lambda} - q^{r-j+1}\mathbf{M}^{j-1}_{\lambda-\sum_{t=j}^{r}\gamma_t},$$

where $j \leq r \leq n-i+1$ is the smallest integer such that $\lambda_r > 0$. If such an integer does not exist then $\mathbf{M}^{j}_{\lambda} = \mathbf{M}^{j-1}_{\lambda}$.

Proof. Let us first assume that the integer r does exist. If $r = j$ then $\lambda - \gamma_j \in X^+$ (recall that by hypothesis $\lambda_{j+i-1} > 0$) and we conclude by eq. (4.2). Thus we assume that $r > j$. We define $B_j = \Gamma_{j-1} \setminus \{\alpha_{j,j+i}\}$ and for $j < k \leq n-i$ we define recursively $B_k = B_{k-1} \setminus \{\alpha_{k,k+i}\}$. At first glance, this definition might seem odd, but B_k is defined in such a way in order to satisfy the equation $s_k(B_k) = B_k$ for every k satisfying $j \leq k < r$ (the proof of that equality is elementary but lengthy, and is left to the reader). If $\mu := \sum_{t=j}^{r}\gamma_t$, we have

$$\begin{aligned}
\mathbf{M}^{j-1}_{\lambda-\mu} &= \mathbf{M}^{B_j}_{\lambda-\mu} - q\mathbf{M}^{B_j}_{\lambda-\mu-\alpha_{j,j+i}}\\
\mathbf{M}^{B_j}_{\lambda-\mu} &= \mathbf{M}^{B_{j+1}}_{\lambda-\mu} - q\mathbf{M}^{B_{j+1}}_{\lambda-\mu-\alpha_{j+1,j+1+i}}\\
\mathbf{M}^{B_{j+1}}_{\lambda-\mu} &= \mathbf{M}^{B_{j+2}}_{\lambda-\mu} - q\mathbf{M}^{B_{j+2}}_{\lambda-\mu-\alpha_{j+2,j+2+i}}\\
\vdots &= \vdots\\
\mathbf{M}^{B_{r-2}}_{\lambda-\mu} &= \mathbf{M}^{B_{r-1}}_{\lambda-\mu} - q\mathbf{M}^{B_{r-1}}_{\lambda-\mu-\alpha_{r-1,r-1+i}}
\end{aligned}$$

We sum all the equations above and after canceling out similar terms we obtain

$$\mathbf{M}^{j-1}_{\lambda-\mu} = \mathbf{M}^{B_{r-1}}_{\lambda-\mu} - q\sum_{k=j}^{r-1}\mathbf{M}^{B_k}_{\lambda-\mu-\alpha_{k,k+i}}.$$

Let k be an integer satisfying $j \leq k < r$. By minimality of r, the k-th coordinate of $\lambda - \mu - \alpha_{k,k+i}$ when written in terms of the fundamental weights is equal to -1. Therefore, as $s_k(B_k) = B_k$ we use Proposition 4.3 to conclude that $\mathbf{M}^{B_k}_{\lambda-\mu-\alpha_{k,k+i}} = 0$. It follows that

$$\mathbf{M}^{j-1}_{\lambda-\mu} = \mathbf{M}^{B_{r-1}}_{\lambda-\mu}. \tag{4.3}$$

Since $\lambda_j = 0$ and $s_j(B_j) = B_j$, using Proposition 4.3 one can prove the second equality:

$$\mathbf{M}^{j-1}_{\lambda-\gamma_j} = \mathbf{M}^{B_j}_{\lambda-\gamma_j} - q\mathbf{M}^{B_j}_{\lambda-\gamma_j-\alpha_{j,j+i}} = \mathbf{M}^{B_j}_{\lambda-\gamma_j} + q\mathbf{M}^{B_j}_{\lambda-\gamma_j-\gamma_{j+1}}. \tag{4.4}$$

Similarly, since $\lambda_k = 0$ and $s_k(B_k) = B_k$ for all $j < k < r$, Proposition 4.3 implies

$$\mathbf{M}^{B_{k-1}}_{\lambda-\sum_{t=j}^{k}\gamma_t} = \mathbf{M}^{B_k}_{\lambda-\sum_{t=j}^{k}\gamma_t} - q\mathbf{M}^{B_k}_{\lambda-\alpha_{k,k+i}-\sum_{t=j}^{k}\gamma_t} = \mathbf{M}^{B_k}_{\lambda-\sum_{t=j}^{k}\gamma_t} + q\mathbf{M}^{B_k}_{\lambda-\sum_{t=j}^{k+1}\gamma_t} \tag{4.5}$$

Therefore, eq. (4.4) and a repeated application of eq. (4.5) yields

$$\mathbf{M}^{j-1}_{\lambda-\gamma_j} = q^{r-j}\mathbf{M}^{B_{r-1}}_{\lambda-\mu} + \sum_{k=j}^{r-1} q^{k-j}\mathbf{M}^{B_k}_{\lambda-\sum_{t=j}^{k}\gamma_t}.$$

Once again, we can use Proposition 4.3 to conclude that $\mathbf{M}^{B_k}_{\lambda-\sum_{t=j}^{k}\gamma_t} = 0$ for all $j \leq k < r$. It follows that

$$\mathbf{M}^{j-1}_{\lambda-\gamma_j} = q^{r-j}\mathbf{M}^{B_{r-1}}_{\lambda-\mu}.$$

Finally, we obtain

$$\mathbf{M}^{j}_{\lambda} = \mathbf{M}^{j-1}_{\lambda} - q\mathbf{M}^{j-1}_{\lambda-\gamma_j} = \mathbf{M}^{j-1}_{\lambda} - q^{r-j+1}\mathbf{M}^{B_{r-1}}_{\lambda-\mu} = \mathbf{M}^{j-1}_{\lambda} - q^{r-j+1}\mathbf{M}^{j-1}_{\lambda-\mu},$$

where the last equality is eq. (4.3). We have proved the lemma when the integer r exists. We now assume r does not exist. This means that $\lambda_k = 0$ for all $j \leq k \leq n-i+1$. Arguing as before, we get

$$\mathbf{M}^{j-1}_{\lambda-\gamma_j} = q^{n-i+1-j}\mathbf{M}^{B_{n-i}}_{\lambda-\nu},$$

where $\nu := \sum_{t=j}^{n-i+1}\gamma_t$. It is again a lengthy but easy problem to prove that $s_{n-i+1}(B_{n-i}) = B_{n-i}$. Furthermore, the $(n-i+1)$-th component of $\lambda-\nu$ when written in terms of the basis of fundamental weights is -1. By applying again Proposition 4.3 we conclude that $\mathbf{M}^{B_{n-i}}_{\lambda-\nu} = 0$. Finally, we have

$$\mathbf{M}^{j}_{\lambda} = \mathbf{M}^{j-1}_{\lambda} - q\mathbf{M}^{j-1}_{\lambda-\gamma_j} = \mathbf{M}^{j-1}_{\lambda} - q^{n-i+2-j}\mathbf{M}^{B_{n-i}}_{\lambda-\nu} = \mathbf{M}^{j-1}_{\lambda},$$

as required. □

We recall a definition given in the introduction.

Definition 4.6. Let $1 \leq i \leq n$. Let $\lambda, \mu \in X^+$. We write $\lambda \geq_i \mu$ if $\lambda - \mu$ can be written as a positive integral linear combination of the elements of Φ^i.

Theorem 4.7. *Let $n \geq 2$ and $n/2+1 \leq i \leq n$. We have the equation*

$$\mathbf{N}^{i+1}_{\lambda} = \sum_{\mu \leq_i \lambda} q^{\frac{1}{i}\,\mathrm{ht}(\lambda-\mu)}\mathbf{N}^{i}_{\mu}$$

for all $\lambda \in X^+$.

Proof. Let $\bar{n} = n - i + 1$ and $\lambda \in X^+$. The hypothesis ensures that $\bar{n} < i$. Let $1 \leq j \leq \bar{n}$. Suppose that there exists $j \leq r \leq \bar{n}$ such that $\lambda - \sum_{t=j}^{r} \gamma_t \in X^+$. In this case we define $R_j(\lambda) := \lambda - \sum_{t=j}^{r} \gamma_t$ where r is minimal with the above property. If such an r does not exist then $R_j(\lambda)$ is not defined. We stress that $R_j(\lambda)$ is defined if and only if $\lambda_{j+i-1} > 0$ and at least one of the integers $\lambda_j, \lambda_{j+1}, \dots, \lambda_{\bar{n}}$ is greater than zero. With these notations Lemma 4.5 can be restated as follows

$$\mathbf{M}_\lambda^{j-1} = \mathbf{M}_\lambda^{j} + q^{\frac{1}{i}\,\mathrm{ht}(\lambda - R_j(\lambda))} \mathbf{M}_{R_j(\lambda)}^{j-1}. \tag{4.6}$$

If k is the maximal integer such that $R_j^k(\lambda)$ is defined then Lemma 4.4 and eq. (4.6) (applied k times) imply

$$\mathbf{M}_\lambda^{j-1} = \sum_{s=0}^{k} q^{\frac{1}{i}\,\mathrm{ht}(\lambda - R_j^s(\lambda))} \mathbf{M}_{R_j^s(\lambda)}^{j}.$$

Since $\mathbf{M}_\lambda^0 = \mathbf{N}_\lambda^{i+1}$ and $\mathbf{M}_\lambda^{\bar{n}} = \mathbf{N}_\lambda^{i}$ we have that $\mathbf{N}_\mu^i$ occurs in the decomposition of $\mathbf{N}_\lambda^{i+1}$ with coefficient $c_\lambda(\mu) q^{\frac{1}{i}\,\mathrm{ht}(\lambda-\mu)}$ where $c_\lambda(\mu)$ is the number of tuples $(a_1, a_2, \dots, a_{\bar{n}}) \in (\mathbb{Z}_{\geq 0})^{\bar{n}}$ such that

$$R_{\bar{n}}^{a_{\bar{n}}} \cdots R_2^{a_2} R_1^{a_1}(\lambda) = \mu.$$

It is clear that if $\mu \not\leq_i \lambda$ then $c_\lambda(\mu) = 0$. Thus in order to prove the Theorem it is enough to show that if $\mu \leq_i \lambda$ then such a sequence exists and it is unique.

Uniqueness. Assume there are two sequences $(a_1, a_2, \dots, a_{\bar{n}})$ and $(a_1', a_2', \dots, a_{\bar{n}}')$ such that

$$R_{\bar{n}}^{a_{\bar{n}}} \cdots R_2^{a_2} R_1^{a_1}(\lambda) = R_{\bar{n}}^{a_{\bar{n}}'} \cdots R_2^{a_2'} R_1^{a_1'}(\lambda). \tag{4.7}$$

By explicitly calculating $\sum_{t=j}^{r} \gamma_t$ in the basis of fundamental weights, we see that $(R_j(\lambda))_k$ (i.e. the k^{th} component in the decomposition of $R_j(\lambda)$ in the fundamental weights) can only differ from λ_k if

$$k \in \{j-1, j, j+1, \dots, \bar{n}\} \cup \{j+i-1, j+i, j+i+1, \dots, n\}.$$

Remark that the union is disjoint because $\bar{n} < i$ and that $(R_j(\lambda))_{j+i-1} = \lambda_{j+i-1} - 1$. Using this we can compare the coefficient of ϖ_i on both sides of eq. (4.7) and obtain $\lambda_i - a_1 = \lambda_i - a_1'$, and therefore $a_1 = a_1'$. Now assume that for some $1 \leq j < \bar{n}$ we have $a_s = a_s'$ for all $1 \leq s \leq j$. Then, by comparing the coefficient of ϖ_{i+j} on both sides of eq. (4.7) we obtain $c - a_{j+1} = c - a_{j+1}'$, where c is the $(i+j)$-th coordinate of $R_j^{a_j} \cdots R_2^{a_2} R_1^{a_1}(\lambda) = R_j^{a_j'} \cdots R_2^{a_2'} R_1^{a_1'}(\lambda)$. Thus $a_{j+1} = a_{j+1}'$. By induction we conclude that both sequences are the same.

Existence. We will prove existence by induction in $\mathrm{ht}(\lambda-\mu)$. In the base case $\mathrm{ht}(\lambda-\mu)=0$ (i.e. $\lambda=\mu$) there is nothing to prove.

Let $\lambda-\mu=\sum_{j=1}^{\bar{n}} m_j\gamma_j$ and suppose that existence is proved for any λ',μ' such that $\mathrm{ht}(\lambda'-\mu')<\mathrm{ht}(\lambda-\mu)$. Let $\lambda\neq\mu$ and let s be minimal such that $m_s>0$. We have

$$\lambda-\sum_{j=s}^{\bar{n}} m_j\gamma_j=\mu. \tag{4.8}$$

Since $\mu\in X^+$ the above equality implies that $\lambda_{s+i-1}>0$ and that at least one of the following integers $\lambda_s,\lambda_{s+1},\dots,\lambda_{\bar{n}}$ is greater than zero. In particular, $R_s(\lambda)$ is defined. We claim that $\mu\leq_i R_s(\lambda)$. Indeed, if

$$R_s(\lambda)=\lambda-\sum_{j=s}^{r}\gamma_j$$

for some $s\leq r\leq\bar{n}$, then the minimality of r implies $\lambda_s=\lambda_{s+1}=\cdots=\lambda_{r-1}=0$. Since μ is dominant, the above equalities and eq. (4.8) imply that $m_j>0$ for all $s\leq j\leq r$ (by observing the s^{th} component, $m_s>0$ and $\lambda_s=0$ imply that $m_{s+1}>0$. Then observe the $(s+1)^{\mathrm{th}}$ component and so on). Hence

$$R_s(\lambda)-\mu=(R_s(\lambda)-\lambda)+(\lambda-\mu)=\sum_{j=s}^{r}(m_j-1)\gamma_j+\sum_{j=r+1}^{\bar{n}} m_j\gamma_j$$

and the claim follows.

We notice that $\mathrm{ht}(R_s(\lambda)-\mu)<\mathrm{ht}(\lambda-\mu)$. Then, by induction hypothesis, there exists a sequence $(b_1,b_2,\dots,b_{\bar{n}})\in\mathbb{Z}_{\geq 0}^{\bar{n}}$ such that

$$R_{\bar{n}}^{b_{\bar{n}}}\cdots R_2^{b_2}R_1^{b_1}(R_s(\lambda))=\mu. \tag{4.9}$$

By eq. (4.8) we have $\lambda_k=\mu_k$ for all $i\leq k<s+i-1$ (notice that substracting γ_j for $s\leq j\leq\bar{n}$ does not affect the k^{th} component for $i\leq k<s+i-1$). Then eq. (4.9) implies $b_1=b_2=\cdots=b_{s-1}=0$. Therefore, the desired sequence is given by $a_s=b_s+1$ and $a_j=b_j$ for $j\neq s$. □

5. Explicit decompositions in types $\tilde{A}_3$ and $\tilde{A}_4$

5.1. *Type* $\tilde{A}_3$

Throughout this section we fix $n=3$. Let $\lambda=a\varpi_1+b\varpi_2+c\varpi_3\in X^+$. Theorem 4.7 gives the decomposition of $\underline{\mathbf{H}}_\lambda=\mathbf{N}_\lambda^4$ in terms of $\{\mathbf{N}_\mu^3\mid\mu\in X^+\}$ (see eq. (1.1)). Furthermore, Corollary 3.7 provides the decomposition of $\mathbf{N}_\lambda^2$ in terms of $\{\mathbf{N}_\mu^1\mid\mu\in X^+\}$. Therefore, to complete the description of all the decompositions we need to explain the

Table 1
Decomposition of $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$ for $\tilde{A}_3$.

Row	a	b	c	Decomposition
1	0	≥ 1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}}$
2	0	0	≥ 0	$\mathbf{N}^3_\lambda$
3	0	1	0	$\mathbf{N}^3_\lambda$
4	0	≥ 2	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$
5	≥ 1	≥ 1	0	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}}$
6	≥ 0	0	0	$\mathbf{N}^3_\lambda$
7	≥ 1	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}}$
8	≥ 1	0	≥ 1	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{13}}$

decomposition of $\mathbf{N}^3_\lambda$ in terms of $\{\mathbf{N}^2_\mu \mid \mu \in X^+\}$. We proceed indirectly by first finding the inverse decomposition. It follows from Definition 2.9 that

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}} \tag{5.1}$$

for $a \geq 1$, $b \geq 2$ and $c \geq 1$. We refer to the above as the generic decomposition and to any other case as non-generic. The following lemma provides the non-generic decompositions for $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$.

Lemma 5.1. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 \in X^+$. In type $\tilde{A}_3$ the non-generic decomposition for $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$ is given in Table 1.*

Proof. The result follows by a case-by-case analysis. We only prove here the decomposition given by Row 1 in Table 1. So we are in the case $a = 0$, $b \geq 1$ and $c \geq 1$. Let $A = \Phi^{\geq 2} \setminus \{\alpha_{12}\}$. Notice that $s_1(A) = A$. By applying Proposition 4.3 we obtain $\mathbf{N}^2_\lambda = \mathbf{M}^A_\lambda - q\mathbf{M}^A_{\lambda-\alpha_{12}} = \mathbf{M}^A_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}}$, which coincides with the decomposition predicted by Row 1 in Table 1. □

Given $\lambda \in X^+$ we define $\hat{\mathbf{N}}^3_\lambda$ as the right hand side of eq. (1.2).

Lemma 5.2. *Let $\lambda \in X^+$. The decomposition of $\mathbf{N}^2_\lambda$ in terms of $\{\hat{\mathbf{N}}^3_\mu \mid \mu \in X^+\}$ is the same as the decomposition in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$.*

Proof. We only need to check that the decomposition of $\mathbf{N}^2_\lambda$ in terms of $\{\hat{\mathbf{N}}^3_\mu \mid \mu \in X^+\}$ coincides with the one given in eq. (5.1) and Table 1. This requires a case-by-case analysis. We leave the details to the reader since in Section 5.2 we treat in full detail the similar but harder situation in type $\tilde{A}_4$. □

Theorem 5.3. *For all $\lambda \in X^+$ we have $\mathbf{N}^3_\lambda = \hat{\mathbf{N}}^3_\lambda$. Therefore, eq. (1.2) provides the decomposition of $\mathbf{N}^3_\lambda$ in terms of $\{\mathbf{N}^2_\mu \mid \mu \in X^+\}$.*

Proof. This is a direct consequence of Lemma 5.2. □

5.2. Type $\tilde{A}_4$

In this section we fix $n = 4$. We stress that $\underline{\mathbf{H}}_\lambda = \mathbf{N}^5_\lambda$ for all $\lambda \in X^+$. Theorem 4.7 provides the decomposition of $\mathbf{N}^5_\lambda$ in terms of $\{\mathbf{N}^4_\mu \mid \mu \in X^+\}$ and of $\mathbf{N}^4_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$. On the other hand, the decomposition of $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^1_\mu \mid \mu \in X^+\}$ is covered by Corollary 3.7. Thus, such as in the previous section, we only need to specify the decomposition of $\mathbf{N}^3_\lambda$ in terms of $\{\mathbf{N}^2_\mu \mid \mu \in X^+\}$.
Let λ, μ in X^+. We write $\mu \preccurlyeq_2 \lambda$ if $\lambda - \mu$ can be written as an integral non-negative linear combination of the elements of $\Phi^{\geq 2}$. Given $\mu \preccurlyeq_2 \lambda$ we denote by $\mathcal{L}_\lambda(\mu)$ the set of all non-negative linear combinations of the elements of $\Phi^{\geq 2}$ equal to $\lambda - \mu$. We denote a linear combination $L \in \mathcal{L}_\lambda(\mu)$ by $L = (l_{ij})$, where $l_{ij} \in \mathbb{N}$ is the coefficient of α_{ij} in L. Given $L = (l_{ij}) \in \mathcal{L}_\lambda(\mu)$ we define its *degree* as

$$\deg(L) := \sum_{1 \leq i < j \leq 4} l_{ij}(\mathrm{ht}(\alpha_{ij}) - 1).$$

On the other hand, we set $\nu_0(L) = \lambda$ and for $k = 1, 2, 3$ we recursively define

$$\nu_k(L) = \nu_{k-1}(L) - \sum_{j-i=k} l_{ij}\alpha_{ij}.$$

We define integers $\nu^j_k(L)$ by the equation $\nu_k(L) = \nu^1_k(L)\varpi_1 + \nu^2_k(L)\varpi_2 + \nu^3_k(L)\varpi_3 + \nu^4_k(L)\varpi_4$.

Definition 5.4. We say that an element $L \in \mathcal{L}_\lambda(\mu)$ is *admissible* if it satisfies the following conditions

- $v_k(L) \in X^+$ for all $1 \leq k \leq 3$;
- If $l_{13} \neq 0$ then $\nu^2_1(L) = 0$;
- If $l_{24} \neq 0$ then $\nu^3_1(L) = 0$;
- If $l_{14} \neq 0$ then $\nu^2_2(L) = \nu^3_2(L) = 0$;

We denote the set of all admissible L by $\mathcal{L}^a_\lambda(\mu)$ and define $r_\lambda(\mu) = \sum_{L \in \mathcal{L}^a_\lambda(\mu)} q^{\deg(L)}$.

Finally, we define

$$\hat{\mathbf{N}}^3_\lambda = \sum_{\mu \preccurlyeq_2 \lambda} r_\lambda(\mu)\mathbf{N}^2_\mu. \tag{5.2}$$

Theorem 5.5. *For all $\lambda \in X^+$ we have $\mathbf{N}^3_\lambda = \hat{\mathbf{N}}^3_\lambda$. Consequently, eq. (5.2) provides the decomposition of $\mathbf{N}^3_\lambda$ in terms of $\{\mathbf{N}^2_\mu \mid \mu \in X^+\}$.*

The proof of Theorem 5.5 will follow the same lines as the proof of Theorem 5.3, this is, we will eventually show that the expansion of $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$ is the same as the expansion in terms of $\{\hat{\mathbf{N}}^3_\mu \mid \mu \in X^+\}$. As in type $\tilde{A}_3$ we have a generic case.

Proposition 5.6. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 \in X^+$. Assume that $a \geq 1$, $b \geq 2$, $c \geq 2$ and $d \geq 1$. Then, we have*

$$\mathbf{N}^2_\lambda = \sum_{J \subset \Phi^2} (-q)^{|J|} \mathbf{N}^3_{\lambda - \Sigma_J} \tag{5.3}$$

$$\mathbf{N}^2_\lambda = \sum_{J \subset \Phi^2} (-q)^{|J|} \hat{\mathbf{N}}^3_{\lambda - \Sigma_J}. \tag{5.4}$$

Proof. The conditions imposed on a, b, c and d, are equivalent to asking that $\lambda - \Sigma_J \in X^+$ for all $J \subset \Phi^2$. Therefore, eq. (5.3) follows easily from Definition 2.9.
We now prove eq. (5.4). We define elements $m_\mu \in \mathbb{Z}[v, v^{-1}]$ by the equation

$$\sum_{J \subset \Phi^2} (-q)^{|J|} \hat{\mathbf{N}}^3_{\lambda - \Sigma_J} = \sum_{\mu \in X^+} m_\mu \mathbf{N}^2_\mu.$$

By eq. (5.2) it is clear that $m_\lambda = 1$ and that $m_\mu = 0$ if $\mu \not\preccurlyeq_2 \lambda$. We fix $\mu \preccurlyeq_2 \lambda$ with $\mu \neq \lambda$.

Assume that $\mathcal{L}^a_\lambda(\mu) \neq \emptyset$. Let $J \subset \Phi^2$. Given $L = (l_{ij}) \in \mathcal{L}_{\lambda - \Sigma_J}(\mu)$ we define $L' = (l'_{ij}) \in \mathcal{L}_\lambda(\mu)$ as $l'_{ij} := l_{ij} + 1$ if $\alpha_{ij} \in J$, and $l'_{ij} := l_{ij}$, otherwise. This defines a map $F : \mathcal{L}_{\lambda - \Sigma_J}(\mu) \mapsto \mathcal{L}_\lambda(\mu)$. Moreover, as $J \subset \Phi^2$ one has that $\deg(L') = \deg(L) + |J|$. Notice that, since $J \subset \Phi^2$, we have that $\nu_k(L) = \nu_k(L')$ for $1 \leq k \leq 3$ and that $l_{13} = l'_{13}$, $l_{24} = l'_{24}$ and $l_{14} = l'_{14}$. It follows that F preserves admissible elements, i.e. it restricts to a map $F : \mathcal{L}^a_{\lambda - \Sigma_J}(\mu) \mapsto \mathcal{L}^a_\lambda(\mu)$ that is clearly injective. The image $F(\mathcal{L}^a_{\lambda - \Sigma_J}(\mu))$ is the set

$$\mathcal{L}^a_\lambda(\mu)^J := \{L = (l_{ij}) \in \mathcal{L}^a_\lambda(\mu) \mid l_{ij} \neq 0 \text{ for all } \alpha_{ij} \in J\}.$$

From this it follows that

$$\sum_{L \in \mathcal{L}^a_\lambda(\mu)^J} q^{\deg(L)} = q^{|J|} r_{\lambda - \Sigma_J}(q). \tag{5.5}$$

On the other hand, by definition of admissibility we have

$$\mathcal{L}^a_\lambda(\mu) = \bigcup_{\alpha \in \Phi^2} \mathcal{L}^a_\lambda(\mu)^{\{\alpha\}}. \tag{5.6}$$

If in eq. (5.6) one takes the q-graded degree on both sides we obtain by eq. (5.5) and the inclusion-exclusion principle:

$$0 = \sum_{J \subset \Phi^2} (-q)^{|J|} r_{\lambda - \Sigma_J}(\mu). \tag{5.7}$$

(We use here that $\mathcal{L}^a_\lambda(\mu)^{J_1} \cap \mathcal{L}^a_\lambda(\mu)^{J_2} = \mathcal{L}^a_\lambda(\mu)^{J_1 \cup J_2}$ for any $J_1, J_2 \subset \Phi^2$.)

The right-hand side of eq. (5.7) is equal to m_μ. Therefore, we have shown that $m_\mu = 0$ if $\mathcal{L}^a_\lambda(\mu) \neq \emptyset$.

We now assume that $\mathcal{L}^a_\lambda(\mu) = \emptyset$. In this case we have $\mathcal{L}^a_{\lambda-\Sigma_J}(\mu) = \emptyset$ for all $J \subset \Phi^2$ (recall that $\mathcal{L}^a_{\lambda-\Sigma_J}(\mu)$ injects in $\mathcal{L}^a_\lambda(\mu)$). We conclude that $m_\mu = 0$ in this case as well. Summing up, we have proved that $m_\mu = \delta_{\lambda,\mu}$ (where $\delta_{\lambda,\mu}$ is a Kronecker delta). By the definition of m_μ this is equivalent to eq. (5.4). $\square$

Before we prove the non-generic decomposition of $\mathbf{N}^2_\lambda$, we record in the following five lemmas some useful computations which are later needed several times. The proofs of all these lemmas are routine computations using Proposition 4.3 and the definition of the relevant $\mathbf{N}$ and $\mathbf{M}$ elements. For this reason we omit the proofs.

Lemma 5.7. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 \in X^+$. Suppose that $a = 0$. Then*

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{34}} + q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}}. \tag{5.8}$$

Remark 5.8. By definition, if $\lambda \in X^+$, we have $\mathbf{N}^k_\lambda = \mathbf{M}^{\Phi^{\geq k}}_\lambda$ for every k. To avoid confusion, we have defined $\mathbf{N}^k_\lambda$ only for $\lambda \in X^+$. On the other hand, $\mathbf{M}^{\Phi^{\geq k}}_\lambda$ is defined for every $\lambda \in X$. This explains why in eq. (5.8) we cannot write, for example, $\mathbf{N}^3_{\lambda-\alpha_{23}}$ instead of $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}}$.

Lemma 5.9. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 \in X^+$ and suppose that $c = 0$. Then*

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}} - q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{24}} + q^3\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{24}}. \tag{5.9}$$

Lemma 5.10. *Let $\lambda = a\varpi_1 + d\varpi_4 \in X^+$. Then:*

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{14}} + q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} - q^3\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{14}-\alpha_{23}}.$$

Lemma 5.11. *Let $\lambda = a\varpi_1 + d\varpi_4 \in X^+$. Then, we have*

$$\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} = \begin{cases} 0, & \textit{if } \min(a,d) = 0; \\ -q^2\underline{\mathbf{H}}_{\lambda-\alpha_{14}}, & \textit{if } \min(a,d) = 1; \\ -q^2\underline{\mathbf{H}}_{\lambda-\alpha_{14}} + q^3\underline{\mathbf{H}}_{\lambda-2\alpha_{14}}, & \textit{otherwise.} \end{cases}$$

Lemma 5.12. *Let $\lambda = a\varpi_1 + d\varpi_4 \in X^+$. Then,*

$$\mathbf{N}^3_\lambda = \begin{cases} \underline{\mathbf{H}}_\lambda - q\underline{\mathbf{H}}_{\lambda-\alpha_{14}}, & \textit{if } \min(a,d) > 0; \\ \underline{\mathbf{H}}_\lambda, & \textit{if } \min(a,d) = 0. \end{cases}$$

Proposition 5.13. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 \in X^+$. In type $\tilde{A}_4$ the non-generic decomposition for $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$ is given in Table 2.*

Table 2
Decomposition of $\mathbf{N}^2_\lambda$ in terms of $\{\mathbf{N}^3_\mu \mid \mu \in X^+\}$ for $\tilde{A}_4$.

Row	a	b	c	d	Decomposition
1	0	≥ 1	≥ 1	0	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}}$
2	0	≥ 2	0	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$
3	0	≥ 1	≥ 2	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{23}-\alpha_{34}}$
4	0	≥ 1	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$
5	0	≥ 2	0	≥ 1	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{24}} - q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}} + q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{24}}$
6	0	1	0	≥ 2	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{24}} + q^4\mathbf{N}^3_{\lambda-\alpha_{24}-\alpha_{14}}$
7	0	1	0	1	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{24}}$
8	0	1	0	0	$\mathbf{N}^3_\lambda$
9	0	0	≥ 1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{34}}$
10	0	0	0	≥ 0	$\mathbf{N}^3_\lambda$
11	≥ 1	1	≥ 2	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{23}-\alpha_{34}}$
12	≥ 1	≥ 2	0	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q^2\mathbf{N}^3_{\lambda-\alpha_{24}} + q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{24}}$
13	≥ 1	1	0	1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q^2\mathbf{N}^3_{\lambda-\alpha_{24}}$
14	≥ 1	1	0	≥ 2	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q^2\mathbf{N}^3_{\lambda-\alpha_{24}} + q^4\mathbf{N}^3_{\lambda-\alpha_{14}-\alpha_{24}}$
15	≥ 1	0	0	1	$\mathbf{N}^3_\lambda - (q^2+q^3)\mathbf{N}^3_{\lambda-\alpha_{14}}$
16	≥ 2	0	0	≥ 2	$\mathbf{N}^3_\lambda - (q^2+q^3)\mathbf{N}^3_{\lambda-\alpha_{14}} + q^5\mathbf{N}^3_{\lambda-2\alpha_{14}}$
17	≥ 1	1	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{34}} + q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$
18	0	0	≥ 2	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{23}-\alpha_{34}}$
19	≥ 1	≥ 2	≥ 1	0	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{12}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$
20	≥ 1	1	≥ 1	0	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{12}} + q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$
21	≥ 1	0	≥ 2	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{13}} - q^2\mathbf{N}^3_{\lambda-\alpha_{23}-\alpha_{34}} + q^3\mathbf{N}^3_{\lambda-\alpha_{34}-\alpha_{13}}$
22	≥ 2	0	1	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{13}} + q^4\mathbf{N}^3_{\lambda-\alpha_{13}-\alpha_{14}}$
23	1	0	1	0	$\mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{13}}$
24	0	0	1	0	$\mathbf{N}^3_\lambda$
25	≥ 1	≥ 1	0	0	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}}$
26	≥ 1	0	0	0	$\mathbf{N}^3_\lambda$
27	≥ 1	≥ 2	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$
28	≥ 1	0	≥ 2	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{34}} - q^2\mathbf{N}^3_{\lambda-\alpha_{13}} + q^3\mathbf{N}^3_{\lambda-\alpha_{13}-\alpha_{34}}$
29	1	0	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{34}} - q^2\mathbf{N}^3_{\lambda-\alpha_{13}}$
30	≥ 2	0	1	≥ 1	$\mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{34}} - q^2\mathbf{N}^3_{\lambda-\alpha_{13}} + q^4\mathbf{N}^3_{\lambda-\alpha_{13}-\alpha_{14}}$
31	1	0	0	≥ 1	$\mathbf{N}^3_\lambda - (q^2+q^3)\mathbf{N}^3_{\lambda-\alpha_{14}}$

Proof. The proof follows by a case-by-case analysis. Notice that Rows 18-31 are symmetric to Rows 2-15. So we need to consider only the first 17 rows. We will make use of the following sets:

$$B := \Phi^{\geq 2} \setminus \{\alpha_{12}, \alpha_{34}\},\ C := \Phi^{\geq 3} \setminus \{\alpha_{13}\} \text{ and } D := \Phi^{\geq 3} \setminus \{\alpha_{24}\}.$$

Notice that $B = s_1(B) = s_4(B)$, $C = s_1(C) = s_3(C)$ and $D = s_2(D) = s_4(D)$.

R 1. We have $\mathbf{N}^2_\lambda = \mathbf{M}^B_\lambda - q\mathbf{M}^B_{\lambda-\alpha_{12}} - q\mathbf{M}^B_{\lambda-\alpha_{34}} + q^2\mathbf{M}^B_{\lambda-\alpha_{12}-\alpha_{34}}$. Since $s_1(B) = s_4(B) =$

B and $a = d = 0$ we can use Proposition 4.3 to conclude that $\mathbf{M}^B_{\lambda-\alpha_{12}} = \mathbf{M}^B_{\lambda-\alpha_{34}} = \mathbf{M}^B_{\lambda-\alpha_{12}-\alpha_{34}} = 0$. Therefore, $\mathbf{N}^2_\lambda = \mathbf{M}^B_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}}$.

R 2. Arguing as in the proof of Row 1 we arrive to $\mathbf{N}^2_\lambda = \mathbf{M}^B_\lambda = \mathbf{N}^3_\lambda - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}}$. Using set C to decompose the elements $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}}$ and $\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$ and applying Proposition 4.3 we obtain $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} = q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$. Therefore, $\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}$.

R 3. This case is a direct consequence of Lemma 5.7.

R 4. By Lemma 5.7 we have

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}}. \tag{5.10}$$

We have $s_3(C) = C$, thus Proposition 4.3 implies that $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}} = q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$. By plugging this in eq. (5.10) we obtain the desired decomposition.

R 5. Lemma 5.7 implies

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{34}} + q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}}. \tag{5.11}$$

Arguing as in the previous case, using the fact that $s_1(C) = s_3(C) = C$, we obtain

$$\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} = q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}}, \quad \mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{34}} = 0, \quad \mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}} = -\mathbf{N}^3_{\lambda-\alpha_{24}} + q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{24}}. \tag{5.12}$$

Then plugging eq. (5.12) in eq. (5.11) we obtain the desired decomposition.

R 6. This case is similar to Row 5.

R 7, R 8. These rows provide a specific value for λ. Therefore, they follow by a direct computation.

R 9. By Lemma 5.7 we have

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{34}} - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}} + q^2\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}}. \tag{5.13}$$

We use the set D to expand the **M**-elements above. Since $s_2(D) = D$, Proposition 4.3 shows that eq. (5.13) reduces to the decomposition predicted by the table.

R 10. Arguing as in the proof of Row 9 we obtain $\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{34}}$. Using Proposition 4.3 and the fact that $s_1(C) = s_3(C) = C$ we obtain $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{34}} = \mathbf{M}^C_{\lambda-\alpha_{34}} - q\mathbf{M}^C_{\lambda-\alpha_{34}-\alpha_{13}} = 0$.

R 11. By definition of $\mathbf{N}^2_\lambda$ we have

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{23}-\alpha_{34}} + q^2 Z,$$

where $Z = \mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$. Using the set D to further decompose Z, since $s_2(D) = D$ we obtain $Z = 0$.

R 12. This is a direct consequence of Lemma 5.9.

R 13. Using Lemma 5.9, it remains to show that $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{24}} = 0$. This follows after decomposing it using D, since $s_2(D) = s_4(D) = D$.

R 14. Using Lemma 5.9, it remains to show that $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{24}} = q\mathbf{N}^3_{\lambda-\alpha_{14}-\alpha_{24}}$. This follows after decomposing both sides using D, since $s_2(D) = D$.
R 15. Using Lemma 5.10 and Lemma 5.11 we obtain $\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{14}} - q^3\underline{\mathbf{H}}_{\lambda-\alpha_{14}}$. We conclude by noticing that Lemma 5.12 implies that $\mathbf{N}^3_{\lambda-\alpha_{14}} = \underline{\mathbf{H}}_{\lambda-\alpha_{14}}$.
R 16. Let us first assume that $\min(a,d) > 2$. Arguing as in Row 15, by Lemmas 5.10 and 5.11 we obtain

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q^2\mathbf{N}^3_{\lambda-\alpha_{14}} - q^3\underline{\mathbf{H}}_{\lambda-\alpha_{14}} + q^4\underline{\mathbf{H}}_{\lambda-2\alpha_{14}} + q^5\underline{\mathbf{H}}_{\lambda-2\alpha_{14}} - q^6\underline{\mathbf{H}}_{\lambda-3\alpha_{14}}. \tag{5.14}$$

Then, Lemma 5.12 implies that $\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - (q^2+q^3)\mathbf{N}^3_{\lambda-\alpha_{14}} + q^5\mathbf{N}^3_{\lambda-2\alpha_{14}}$, which is the desired decomposition. Assume now $\min(a,d) = 2$. In this case, the term $q^6\underline{\mathbf{H}}_{\lambda-3\alpha_{14}}$ does not appear in eq. (5.14). However, Lemma 5.12 also implies the desired decomposition in this case.
R 17. By definition of $\mathbf{N}^2_\lambda$ we have

$$\mathbf{N}^2_\lambda = \mathbf{N}^3_\lambda - q\mathbf{N}^3_{\lambda-\alpha_{12}} - q\mathbf{N}^3_{\lambda-\alpha_{23}} - q\mathbf{N}^3_{\lambda-\alpha_{34}} + q^2\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{34}} - q^3\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}} + q^2Y, \tag{5.15}$$

where $Y = \mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{23}} + \mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}}$. Using the set D, we obtain $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{12}-\alpha_{23}} = q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$. By symmetry, we also have $\mathbf{M}^{\Phi^{\geq 3}}_{\lambda-\alpha_{23}-\alpha_{34}} = q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$. Therefore, $Y = 2q\mathbf{N}^3_{\lambda-\alpha_{12}-\alpha_{23}-\alpha_{34}}$ and eq. (5.15) reduces to the desired decomposition.

Having checked all 17 cases, the proof is complete. □

We now move to proving "hat" version of the proposition above. As before, we start by recording in the following Lemmas some useful computations. Both lemmas follow using the definition of admissibility. We leave the proofs to the reader.

Lemma 5.14. *Let $\lambda = a\varpi_1 + d\varpi_4 \in X^+$. Then,*

$$\hat{\mathbf{N}}^3_\lambda = \sum_{j=0}^{\min(a,d)} \left(q^{2j} \sum_{k=0}^{j} q^k \right) \mathbf{N}^2_{\lambda-j\alpha_{14}}.$$

Lemma 5.15. *Let $\lambda = a\varpi_1 + b\varpi_2 + d\varpi_4 \in X^+$.*

1. If $a, b \geq 1$ then

$$\hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}} = \begin{cases} \displaystyle\sum_{i=0}^{d} q^{2i}\mathbf{N}^2_{\lambda-i\alpha_{24}} & \text{if } b \geq d; \\ \displaystyle\sum_{i=0}^{b} q^{2i}\mathbf{N}^2_{\lambda-i\alpha_{24}} + \sum_{j=1}^{\min(a+b,d-b)} q^{2b+3j}\mathbf{N}^2_{\lambda-b\alpha_{24}-j\alpha_{14}}, & \text{if } b < d. \end{cases} \tag{5.16}$$

2. *If $a = 0$, $b \geq 2$ and $d \geq 1$ then $\hat{\mathbf{N}}^3_\lambda - q^2\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}-\alpha_{23}}$ is equal to the right-hand side of eq. (5.16).*

Proposition 5.16. *Let $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 \in X^+$. In type $\tilde{A}_4$ the non-generic decomposition for $\mathbf{N}^2_\lambda$ in terms of $\{\hat{\mathbf{N}}^3_\mu \mid \mu \in X^+\}$ is given in Table 2 (replacing $\mathbf{N}^3_\mu$ with $\hat{\mathbf{N}}^3_\mu$).*

Proof. As in the proof of Proposition 5.13 we only need to consider the first 17 rows of Table 2. We first notice that the argument given in the proof of Proposition 5.6 carries over for the decomposition in Rows 1, 2, 3, 4, 9, 11 and 17. On the other hand, Rows 7 and 8 provide a specific value for λ and therefore these cases follow by a direct computation. Furthermore, Lemma 5.14 gives us the result for Rows 10, 15 and 16. This leaves us with five cases to be checked.

R 5. Using Lemma 5.15(2) for λ and Lemma 5.15(1) for $\lambda - \alpha_{24}$ we obtain, as required:

$$\mathbf{N}^2_\lambda = \left(\hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}-\alpha_{23}}\right) - q^2\left(\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}} - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}-\alpha_{12}}\right).$$

R 6. Using the definition of $\hat{\mathbf{N}}^3_\lambda$ and noticing which μ are such that $\mathcal{L}^a_\lambda(\mu) \neq \emptyset$, one obtains

$$\hat{\mathbf{N}}^3_\lambda = \mathbf{N}^2_\lambda + q^2\mathbf{N}^2_{\lambda-\alpha_{24}} + q^5\mathbf{N}^2_{\lambda-\alpha_{24}-\alpha_{14}}. \tag{5.17}$$

On the other hand, using Lemma 5.14 twice, we obtain

$$\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}} - q^2\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}-\alpha_{14}} = \mathbf{N}^2_{\lambda-\alpha_{24}} + q^3\mathbf{N}^2_{\lambda-\alpha_{24}-\alpha_{14}}. \tag{5.18}$$

Combining eq. (5.17) with eq. (5.18) we get the desired decomposition.

R 12. Using Lemma 5.15(1) for λ and $\lambda - \alpha_{24}$ we obtain the required decomposition

$$\mathbf{N}^2_\lambda = \left(\hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}}\right) - q^2\left(\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}} - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}-\alpha_{12}}\right).$$

R 13. By Lemma 5.15(1) we have $\hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}} = \mathbf{N}^2_\lambda + q^2\mathbf{N}^2_{\lambda-\alpha_{24}}$. Since $\lambda - \alpha_{24} = (a+1)\varpi_1$, Lemma 5.14 implies $\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}} = \mathbf{N}^2_{\lambda-\alpha_{24}}$. We conclude that $\mathbf{N}^2_\lambda = \hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}} - q^2\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}}$, as we wanted to show.

R 14. Using Lemma 5.15(1) we obtain

$$\hat{\mathbf{N}}^3_\lambda - q\hat{\mathbf{N}}^3_{\lambda-\alpha_{12}} = \mathbf{N}^2_\lambda + q^2 \sum_{j=0}^{\min(a+1,d-1)} q^{3j}\mathbf{N}^2_{\lambda-\alpha_{24}-j\alpha_{14}}. \tag{5.19}$$

On the other hand, Lemma 5.14 implies

$$\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}} - q^2\hat{\mathbf{N}}^3_{\lambda-\alpha_{24}-\alpha_{14}} = \sum_{j=0}^{\min(a+1,d-1)} q^{3j}\mathbf{N}^2_{\lambda-\alpha_{24}-j\alpha_{14}}. \tag{5.20}$$

By combining eq. (5.19) with eq. (5.20) we obtain the desired decomposition.

This finishes the proof of the Proposition. □

Proof of Theorem 5.5. The result follows by combining Proposition 5.6, Proposition 5.13 and Proposition 5.16. □

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