

THE ANTI-SPHERICAL CATEGORY

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ABSTRACT. We study a diagrammatic categorification (the “anti-spherical category”) of the anti-spherical module for any Coxeter group. We deduce that Deodhar’s (sign) parabolic Kazhdan-Lusztig polynomials have non-negative coefficients, and that a monotonicity conjecture of Brenti’s holds. The main technical observation is a localization procedure for the anti-spherical category, from which we construct a “light leaves” basis of morphisms. Our techniques may be used to calculate many new elements of the p -canonical basis in the anti-spherical module.

1. INTRODUCTION

1.1. Kazhdan-Lusztig polynomials are remarkable polynomials associated to pairs of elements in a Coxeter group. They describe the base change matrix between the standard and the Kazhdan-Lusztig basis of the Hecke algebra. Since their discovery by Kazhdan and Lusztig in 1979, these polynomials have found applications throughout representation theory.

A fascinating aspect of the theory is that these polynomials are elementary to define and compute, however they also have deep properties that are far from obvious from their definition. For example, it was conjectured by Kazhdan and Lusztig in [KL79] that these polynomials have non-negative coefficients. This conjecture was established soon after by Kazhdan and Lusztig [KL80] if the underlying Coxeter group is a finite or an affine Weyl group. Kazhdan and Lusztig’s conjecture was established in complete generality by Elias and the second author via Soergel bimodule techniques [EW14].

In 1987 Deodhar introduced parabolic Kazhdan-Lusztig polynomials [Deo87]. These polynomials are defined starting from the choice of a Coxeter group, a standard parabolic subgroup and a sign. They describe the base change matrix between the standard and Kazhdan-Lusztig basis of the spherical or anti-spherical (depending on the sign) module for the Hecke algebra. Kazhdan-Lusztig polynomials agree with parabolic Kazhdan-Lusztig polynomials for the choice of the trivial parabolic subgroup. Parabolic Kazhdan-Lusztig polynomials are also known to have deep representation theoretic and geometric significance. One of the two main theorems of this paper is the following:

Theorem 1.1. *Parabolic Kazhdan-Lusztig polynomials associated to the sign representation have non-negative coefficients, for any Coxeter system and any choice of standard parabolic subgroup.*

Two remarks on this theorem:

- (1) Kazhdan and Lusztig have a theorem that identifies Kazhdan-Lusztig polynomials with the Poincaré polynomials of the stalks of intersection cohomology complexes on the flag variety. The parabolic analogue of that theorem was given in a beautiful paper by Kashiwara and Tanisaki [KT02] in 2002. Thus the above theorem was already known for the case where both the Coxeter group arises as the Weyl group of a symmetrisable Kac-Moody Lie algebra (this is the case if and only if the order of the product of any two simple reflections belongs to the set $\{2, 3, 4, 6, \infty\}$) and the standard parabolic subgroup is finite.
- (2) The analogue of this theorem for the trivial representation is known if the standard parabolic subgroup is finite. This is because parabolic Kazhdan-Lusztig polynomials associated to the trivial representation are special cases of ordinary Kazhdan-Lusztig polynomials (see section 2.7 for details). We believe that the methods of this paper can be adapted to deduce the analogue of this theorem for parabolic Kazhdan-Lusztig polynomials associated to the trivial representation without the finiteness condition.

1.2. The proof that Kazhdan-Lusztig polynomials have non-negative coefficients in [EW14] relies on a detailed study of a categorification of the Hecke algebra via certain bimodules constructed by Soergel [Soe90, Soe07], which have come to be known as Soergel bimodules. The essential point (“Soergel’s conjecture”) is that the Kazhdan-Lusztig basis arises as the classes in the Grothendieck group of indecomposable Soergel bimodules. Thus Soergel bimodules provide a setting where Kazhdan-Lusztig polynomials have an interpretation as graded dimensions of certain Hom spaces.

More recently, Elias and the second author described the monoidal category of Soergel bimodules by generators and relations [EW16]. The result is a diagrammatically defined additive graded monoidal category which is equivalent to the monoidal category of Soergel bimodules. In this paper we work almost exclusively with this category, which we denote \mathcal{H} and call the Hecke category.

It is natural to try to understand parabolic Kazhdan-Lusztig polynomials by categorifying the modules in which they live. This is precisely what we do in this paper for the anti-spherical module.

1.3. Let (W, S) be a Coxeter system, and let H be its Hecke algebra over $\mathbb{Z}[v^{\pm 1}]$. Let h_x denote its standard basis and b_x its canonical (or Kazhdan-Lusztig) basis. Fix a subset $I \subset S$ and let ${}^I W$ denote the set of minimal coset representatives for $W_I \backslash W$. Let N denote the anti-spherical (right) H -module

$$N := \text{sgn}_v \otimes_{H_I} H,$$

where sgn_v denotes the quantized sign representation of H_I , the standard parabolic subalgebra of H determined by I . Let n_x denote the standard basis of N and d_x its Kazhdan-Lusztig basis.

Recall the Hecke category \mathcal{H} from above. For any $w \in W$ there exists an indecomposable self-dual object $B_w \in \mathcal{H}$ parametrized by w . Any indecomposable self-dual object in \mathcal{H} is isomorphic to B_w for some $w \in W$. We have a canonical isomorphism of $\mathbb{Z}[v^{\pm 1}]$ -algebras

$$H \xrightarrow{\sim} [\mathcal{H}]$$

defined on generators by $b_s \mapsto [B_s]$, for all $s \in S$. Here we have employed the following notation: given an additive graded (with shift functor $M \mapsto M(1)$) category \mathcal{M} , let $[\mathcal{M}]$ denote its split Grothendieck group, which we view as a $\mathbb{Z}[v^{\pm 1}]$ -module via $v[M] := [M(1)]$. Note that $[\mathcal{H}]$ is an algebra because \mathcal{H} is a monoidal category.

Now inside \mathcal{H} consider \mathcal{I} the additive category consisting of all direct sums of shifts of B_x , for $x \notin {}^I W$. It turns out that \mathcal{I} is a right tensor ideal of \mathcal{H} (i.e. if $X \in \mathcal{I}$ and $B \in \mathcal{H}$ then $XB \in \mathcal{I}$). In particular, if we consider the quotient¹ of additive categories

$$\mathcal{N} := \mathcal{H}/\mathcal{I}$$

then this is a right module category over \mathcal{H} . We call \mathcal{N} the *anti-spherical category* (associated to the subset $I \subset S$). The following theorem justifies the name:

Theorem 1.2. *There is a canonical isomorphism $N \xrightarrow{\sim} [\mathcal{N}]$ of $\mathbb{Z}[v^{\pm 1}]$ -modules. This is an isomorphism of right H -modules via the identification $H = [\mathcal{H}]$. Under this isomorphism, the indecomposable self-dual objects in \mathcal{N} correspond to the Kazhdan-Lusztig basis in N .*

We also prove a theorem giving a (“light leaves”) basis for the morphisms between certain additive generators of \mathcal{N} (see Theorem 5.3). From this the positivity of the corresponding parabolic Kazhdan-Lusztig polynomials (Theorem 1.1) is an easy consequence. We also deduce (see Corollary 6.4) from these results a proof of a conjecture of Brenti [Mon14] on the monotonicity of parabolic Kazhdan-Lusztig polynomials associated to increasing subsets $I \subseteq J \subseteq S$.

1.4. We were also motivated in our study of the anti-spherical category by representation theory. If W is the Weyl group of a complex semi-simple Lie algebra, the anti-spherical category can be used to give a graded deformation of parabolic category \mathcal{O} (the subset $I \subset S$ is determined by the parabolic subgroup appearing in the definition of parabolic category \mathcal{O}). This fact does not seem to be available explicitly in the literature, however the papers [Str05] and [KMS08] contain results which are quite close.

The anti-spherical category is also important in modular representation theory. Riche and the second author conjectured that the Hecke category acts via translation functors on the principal block of representations of an algebraic group [RW18]. This conjecture was proved in [RW18] for GL_n , and has recently been established in general by Bezrukavnikov-Riche [BR20] and Ciappara [Cia21]. Thus the anti-spherical category sees all of the (extremely subtle) representation theory of connected reductive algebraic groups. (These developments were heavily motivated by earlier work of Soergel [Soe97] and Arkhipov-Bezrukavnikov [AB09].) In a parallel development, Elias and Losev [EL] explained that one can use singular Soergel bimodules to construct the categories of polynomial representations of GL_n together with the action of certain natural endofunctors, in a purely combinatorial way. Their work provides further evidence for the importance of the anti-spherical category in modular representation theory.

In [RW18] (the obvious analogue of) Theorem 1.2 is proved for the anti-spherical module of an affine Weyl group. (The parabolic subgroup is taken to be the finite Weyl group.) The proofs there rely on geometry or representation theory in a

¹By quotient we mean the following: the objects of \mathcal{N} are the same as those of \mathcal{H} ; a morphism is zero in \mathcal{N} if and only if it factors through an object of \mathcal{I} .

crucial way. One of the main motivations for the current work was to give purely algebraic proofs of these basic statements, which work for any Coxeter system. The proofs of the current paper involve quite different technology than those of [RW18] and are simpler and more general.

1.5. Another consequence of the conjectures of [RW18] is a character formula for simple modules and indecomposable tilting modules for reductive algebraic groups in characteristic p in terms of the p -canonical basis of the anti-spherical module. This conjecture was first proved by Achar, Makisumi, Riche and the second author [AMRW, AMRW19], and has recently be proved in greater generality² by Riche and the second author [RW20]. The paper [EL] of Elias and Losev has related results for GL_n .

The upshot is that the p -canonical basis in the anti-spherical module contains the answers to several deep mysteries in the representation theory of algebraic groups. However it is still not easy to compute. The third main theorem of this paper (Theorem 3.4) heuristically says that the localisation of the anti-spherical category is “as simple as possible”. This was unexpected for the authors because general cell quotients of the Hecke category can have complicated endomorphism rings (for a detailed example, see Elias’ Temperley-Lieb quotient of the Hecke algebra [Eli10]).

Theorem 3.4 is the technical heart of the paper. It also forms the foundation for an effective algorithm to calculate the p -canonical basis. The basic idea is that via localisation one can reduce calculations of the p -canonical basis in the anti-spherical module (which can be performed via diagrammatics, as explained in [JW17]) to certain linear algebra problems over a polynomial ring (the ring denoted R_I in §3.7). In the special case of an affine Weyl group, with parabolic subgroup the finite Weyl group, the ring R_I has one variable, but in general it might have several variables. This algorithm has been further developed and implemented by the second author, Jensen and Gibson to provide a powerful new means to calculate characters of tilting modules, and hence decomposition numbers for symmetric groups [GJW]. This produced new data that was key for the production of the “billiards conjecture” by Lusztig and the second author [LW18].

1.6. We conclude this introduction with a remark on positive characteristic. In the body of this paper we work over a field of characteristic zero. This is because our results rely crucially on the so-called parabolic property of root systems (see (2.3)), which often fails for reflection representations of Coxeter groups in positive characteristic. The parabolic property ensures that Theorem 3.4 holds. It is an interesting question as to what happens if one localises in settings in which the parabolic property fails (as is the case for the important example of the natural representation of affine Weyl groups in characteristic p). At the time this paper was written, this question was mysterious to the authors. However, in the meantime there have been considerable advances in understanding this question, both from the algebraic side in the work of Hazi [Haz18], and on the geometric side via Smith-Treumann theory [Tre19, LL21, RW20]. The relations between these two theories, as well as why Smith-Treumann theory is relevant for describing localizations of the Hecke category in characteristic p is explained in [Wil20].

Finally, let us remark that one can still apply the techniques of this paper to settings in positive characteristic by using the p -adic integers in place of a field of

²i.e. for all weights, and all p

characteristic p . This is one of the basic ideas in the algorithm mentioned in the last paragraph.

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1.8. Note to the reader: A previous version of this article (available on the arxiv) took a significantly more complicated route to our main theorem, by exploiting properties of the infinite twist. This approach contained gaps, pointed out by two referees. Whilst we believe that our original approach still works, the referee's questions lead us to the simplified proof presented here. We are very grateful to both referees. We remain interested in the possibilities of the infinite twist, but omit discussion of it here. The authors learned much about the infinite twist from discussions with M. Hogencamp.

2. PARABOLIC KAZHDAN-LUSZTIG POLYNOMIALS

2.1. The Hecke algebra. We follow the notation of [Soe97]. Let (W, S) be a Coxeter system and $(m_{sr})_{s,r \in S}$ its Coxeter matrix. Let $l : W \rightarrow \mathbb{N}$ be the corresponding length function and \leq the Bruhat order on W . Let $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$ be the ring of Laurent polynomials with integer coefficients in one variable v .

The *Hecke algebra* $H = H(W, S)$ of a Coxeter system (W, S) is the associative algebra over \mathcal{L} with generators $\{h_s\}_{s \in S}$, quadratic relations $(h_s + v)(h_s - v^{-1}) = 0$ for all $s \in S$, and braid relations $h_s h_r h_s \cdots = h_r h_s h_r \cdots$ with m_{sr} elements on each side for every couple $s, r \in S$.

Consider $x \in W$. To a reduced expression $sr \cdots t$ of x one can associate the element $h_s h_r \cdots h_t \in H$. It was proved by H. Matsumoto that this element is independent of the choice of reduced expression of x , and we call it h_x . N. Iwahori proved that

$$H = \bigoplus_{x \in W} \mathcal{L} h_x,$$

and $h_x h_y = h_{xy}$ if $l(x) + l(y) = l(xy)$ (which is clear by Matsumoto's Theorem).

Let us define the element $b_s = h_s + v$. The right regular action of H is given by the formula:

$$(2.1) \quad h_x b_s = \begin{cases} h_{xs} + v h_x & \text{if } x < xs; \\ h_{xs} + v^{-1} h_x & \text{if } x > xs. \end{cases}$$

2.2. Parabolic subgroups. Consider $I \subset S$ an arbitrary subset and W_I its corresponding Coxeter group, which identifies naturally as a subgroup of W . We say that W_I is the *parabolic subgroup* corresponding to I . We say that a sequence \underline{w} of elements in S is an *I-sequence* if it starts with some element $s \in I$.

We denote by ${}^I W \subseteq W$ the set of minimal coset representatives in $W_I \backslash W$. The following two descriptions of this set will be useful for us:

$$(2.2) \quad {}^I W = \{w \in W \mid sw > w \text{ for all } s \in I\};$$

$$(2.3) \quad {}^I W = \{w \in W \mid \text{no reduced expression of } w \text{ is an } I\text{-sequence}\}.$$

Example 2.1. Let W be the symmetric group $W = S_8$ with simple reflections s_1, s_2, \dots, s_7 . For simplicity we will just denote s_k by k , so by 343 we mean the element $s_3 s_4 s_3 \in W$. Let us define the set

$$\underline{54321} := \{\emptyset, 5, 54, 543, 5432, 54321\} \subseteq W.$$

We define in the same way the set $\underline{k \dots 321}$ for any natural number k . The order of this set is $k + 1$.

Say that $I = \{1, 2, 3\}$. Then W_I and ${}^I W$ are the following products of sets

$$W_I = \underline{1} \underline{21} \underline{321} \quad (\text{it has order } 2 \cdot 3 \cdot 4 = 24) \text{ and}$$

$${}^I W = \underline{4321} \underline{54321} \underline{654321} \underline{7654321} \quad (\text{it has order } 5 \cdot 6 \cdot 7 \cdot 8 = 1680).$$

For example, $12132 \in W_I$ and $435436765432 \in {}^I W$.

We see in this example (if one recalls the normal form of an element in the symmetric group) that multiplication defines an isomorphism of sets

$$(2.4) \quad W \cong W_I \times {}^I W$$

satisfying that, if $x \in W_I$ and $y \in {}^I W$, then $l(xy) = l(x) + l(y)$. Deodhar [Deo87] proved that this is true for any Coxeter system and any parabolic subgroup.

2.3. Parabolic Property. Let \mathfrak{h} be the ‘‘dual geometric representation’’ of W (see Section 3.1). Let $\Delta_I := \{\alpha_r\}_{r \in I} \subset \mathfrak{h}^*$ and let $\Phi_I := W_I \cdot \Delta_I$ be the root system spanned by Δ_I . Another important property of minimal coset representatives is what we will call the *Parabolic Property*:

If $x \in {}^I W$ and $s \in S$, then

$$xs \notin {}^I W \iff x(\alpha_s) \in \Delta_I \iff x(\alpha_s) \in \Phi_I.$$

Proof.

- We first prove that $xs \notin {}^I W \Rightarrow x(\alpha_s) \in \Delta_I$. If $x \in {}^I W$ and $xs \notin {}^I W$ then $xs = rx$ for some $r \in I$. This comes from the more general (and beautiful) fact [Soe97, §3] that if x is any element of W and $s, r \in S$, the two inequalities $rx > x$ and $rxs < xs$ imply that $rxs = x$. Hence $x(\alpha_s) = rxs(\alpha_s)$. Thus we obtain the equality $r(x(\alpha_s)) = -x(\alpha_s)$. This implies that $x(\alpha_s) = \alpha_r$ or $x(\alpha_s) = -\alpha_r$. The latter equality can not be true:
 - As $xs = rx$ with $r \in I$, we have that $xs > x$ (see 2.2).
 - The proposition in [Hum90, §5.7] states that $x(\alpha_s)$ is a positive root if and only if $xs > x$.
- The proof that $x(\alpha_s) \in \Delta_I \Rightarrow x(\alpha_s) \in \Phi_I$ is trivial.
- Finally we prove that $x(\alpha_s) \in \Phi_I \Rightarrow xs \notin {}^I W$. As $x(\alpha_s) \in \Phi_I$, we know by the Lemma in [Hum90, §5.7] that $xsx^{-1} = t \in W_I$ with t the reflection satisfying $x(\alpha_s) = \alpha_t$. Rewriting this equation we have $xs = tx$. Bijection 2.4 implies that $xs \notin {}^I W$. \square

2.4. Spherical and anti-spherical modules. We base the exposition and notations of the next sections in [Soe97]. Consider $I \subset S$ and the Hecke algebra $H_I := H(W_I, I)$. By the relations defining the Hecke algebra, if we fix $u \in \{-v, v^{-1}\}$, we can define a surjection of \mathcal{L} -algebras

$$\varphi_u : H_I \twoheadrightarrow \mathcal{L}$$

by sending $h_s \mapsto u$ for all $s \in I$. Thus \mathcal{L} becomes an H_I -bimodule which we denote by $\mathcal{L}(u)$. We can induce from it to produce the following right H -modules:

$$N = N(W, S, I) = \mathcal{L}(-v) \otimes_{H_I} H, \text{ the anti-spherical module;}$$

$$M = M(W, S, I) = \mathcal{L}(v^{-1}) \otimes_{H_I} H, \text{ the spherical module.}$$

If $n_x := 1 \otimes h_x \in N$ and $m_x := 1 \otimes h_x \in M$, then we have that

$$N = \bigoplus_{x \in {}^I W} \mathcal{L}n_x \quad \text{and} \quad M = \bigoplus_{x \in {}^I W} \mathcal{L}m_x.$$

We will not prove this result but we will explain why it is reasonable. Equality (2.2) tells us that if $x \notin {}^I W$, then there is $r \in I$ such that $rx < x$, so $n_x = -vn_{rx}$. In this way we see that the set $\{n_x\}_{x \in {}^I W}$ generates N over \mathcal{L} (a similar result holds for M).

2.5. Right action of the Hecke algebra. The right action of H on the anti-spherical and on the spherical modules (compare with the regular action (2.1)) is given by the formulas

$$(2.5) \quad n_x b_s = \begin{cases} n_{xs} + vn_x & \text{if } x < xs \text{ and } xs \in {}^I W; \\ n_{xs} + v^{-1}n_x & \text{if } x > xs \text{ and } xs \in {}^I W; \\ 0 & \text{if } xs \notin {}^I W. \end{cases}$$

$$(2.6) \quad m_x b_s = \begin{cases} m_{xs} + vm_x & \text{if } x < xs \text{ and } xs \in {}^I W; \\ m_{xs} + v^{-1}m_x & \text{if } x > xs \text{ and } xs \in {}^I W; \\ (v + v^{-1})m_x & \text{if } xs \notin {}^I W. \end{cases}$$

Let us explain these formulas for the anti-spherical module. Similar arguments work in the spherical case. The first two equations of (2.5) are an easy consequence of (2.1). The third equation of (2.5) is a consequence of the following three facts:

- (a) $\varphi_{-v}(b_s) = 0$ for $s \in I$.
- (b) If $x \in {}^I W$ and $xs \notin {}^I W$ then $xs = rx$ for some $r \in I$.
- (c) If $x \in {}^I W$ and $xs \notin {}^I W$ then $xs > x$.

Fact (a) is trivial. We have already seen facts (b) and (c) in §2.3.

2.6. Kazhdan-Lusztig bases. There is a unique ring homomorphism $h \mapsto \bar{h}$ on H such that $\bar{v} = v^{-1}$ and $\bar{h}_x = (h_{x^{-1}})^{-1}$. Recall that $b_s = h_s + v$. If $s \in I$ we have that $\varphi_{-v}(b_s) = 0$ and $\varphi_{v^{-1}}(b_s) = (v + v^{-1})$. In any case $\varphi_u(\bar{b}_s) = \overline{\varphi_u(b_s)}$ so, since the set $\{b_s\}_{s \in S}$ generates H_I as an \mathcal{L} -algebra, we have

$$(2.7) \quad \varphi_u(\bar{h}_I) = \overline{\varphi_u(h_I)} \text{ for any element } h_I \in H_I.$$

We also denote by $\overline{(-)}$ the involution of \mathcal{L} given by $v \mapsto v^{-1}$. Using equation (2.7), we can induce the morphism $\overline{(-)}$ to a morphism of additive groups $\overline{(-)} : N \rightarrow N$ given by $l \otimes h \mapsto \bar{l} \otimes \bar{h}$. In the same way we can induce a morphism of additive groups $\overline{(-)} : M \rightarrow M$. We will call an element *self-dual* if it is invariant under $\overline{(-)}$.

We can now state the central theorem of Kazhdan-Lusztig theory and its parabolic versions.

- Theorem 2.2.**
- (1) ([KL79]) *For every element $x \in W$ there is a unique self-dual element $b_x \in H$, such that $b_x \in h_x + \sum_{y \in W} v\mathbb{Z}[v]h_y$.*
 - (2) ([Deo87]) *For every element $x \in {}^I W$ there is a unique self-dual element $c_x \in M$, such that $c_x \in m_x + \sum_{y \in {}^I W} v\mathbb{Z}[v]m_y$.*

- (3) ([Deo87]) For every element $x \in {}^I W$ there is a unique self-dual element $d_x \in N$, such that $d_x \in n_x + \sum_{y \in {}^I W} v\mathbb{Z}[v]n_y$.

The sets $\{b_x\}_{x \in W}$, $\{c_x\}_{x \in {}^I W}$ and $\{d_x\}_{x \in {}^I W}$ are bases of the corresponding H -modules, and are called the *Kazhdan-Lusztig bases*. For each couple of elements $x, y \in W$ we define $h_{y,x} \in \mathcal{L}$ by the formula

$$b_x = \sum_y h_{y,x} h_y.$$

For each couple of elements $x, y \in {}^I W$ we define $m_{y,x} \in \mathcal{L}$ and $n_{y,x} \in \mathcal{L}$ by the formulae

$$c_x = \sum_{y \in {}^I W} m_{y,x} m_y \quad \text{and} \quad d_x = \sum_{y \in {}^I W} n_{y,x} n_y.$$

(If we need to specify the set I , we will write $m_{y,x}^I$ for $m_{y,x}$ and $n_{y,x}^I$ for $n_{y,x}$.) The proof of Theorem 2.2 (as given by Soergel in [Soe97]) is short and easy. It constructs the Kazhdan-Lusztig basis inductively on the length of x .

The Kazhdan-Lusztig polynomials (as defined in [KL79]) are given by the formula $P_{y,x} = (v^{l(y)-l(x)})h_{y,x}$ and they are polynomials in $q := v^{-2}$. The same normalization gives Deodhar's parabolic polynomials. More precisely $(v^{l(y)-l(x)})m_{y,x}$ and $(v^{l(y)-l(x)})n_{y,x}$ are the polynomials $P_{y^{-1},x^{-1}}^I$ defined by Deodhar in [Deo87] in the cases $u = -1$ and $u = q$, respectively.

2.7. Some relations between these polynomials.

- (1) In the case $I = \emptyset$ we have $H = M = N$, $b_x = c_x = d_x$ and $h_{y,x} = m_{y,x} = n_{y,x}$. Thus the theory of parabolic Kazhdan-Lusztig polynomials contains the theory of Kazhdan-Lusztig polynomials.
- (2) If I is finitary (i.e. W_I is finite) then Deodhar [Deo87] proves that the m polynomials are instances of Kazhdan-Lusztig polynomials. More precisely, he proves that if w_0 is the longest element of W_I then $m_{y,x} = h_{w_0 y, w_0 x}$. Moreover, M is a sub- H -module of H compatible with the duality.

This result was expected. Parabolic Kazhdan-Lusztig polynomials calculate (and this is their main reason to exist) the dimensions of the intersection cohomology modules of Schubert varieties in G/P where G is a Kac-Moody group and P is a standard parabolic. Kazhdan-Lusztig polynomials calculate those dimensions in the case of the flag variety G/B . When G is a semi-simple or affine Kac-Moody group (and thus the parabolic subgroup of the Weyl group of G corresponding to P is finite) one problem reduces to the other, because one has a smooth fibration $G/B \rightarrow G/P$.

- (3) For arbitrary I and $x, y \in {}^I W$, Deodhar [Deo87] proved the formula

$$n_{y,x} = \sum_{z \in W_I} (-v)^{l(z)} h_{zy,x}.$$

This follows from the facts that, if π is the obvious surjection $\pi : H \twoheadrightarrow N$ and $w = xy$ is the decomposition with $x \in W_I$ and $y \in {}^I W$, then we have

$$\pi(h_w) = (-v)^{l(x)} n_y \quad \text{and} \quad \pi(b_y) = d_y.$$

So, summarizing, M is sometimes a good sub-object and N is always a good quotient of H (seen as an H -module).

- (4) If $J \subseteq I$, then for all $y, x \in {}^I W$ $n_{y,x}^I \leq n_{y,x}^J$ (where \leq denotes coefficientwise inequality). This is known as *Brenti's monotonicity conjecture*. This conjecture was stated by Francesco Brenti in 2008 at the Conference “Festive Combinatorics, Symposium in honor of Anders Björner’s 60th Birthday”. We prove it in this paper (see Corollary 6.4) as a consequence of our main theorem.

3. THE CATEGORIES \mathcal{H} , \mathcal{N} AND ${}_{\mathcal{Q}}\mathcal{N}$

In this section we define the Hecke category (denoted by \mathcal{H}), the diagrammatic anti-spherical category (denoted by \mathcal{N}) and a localization (denoted by ${}_{\mathcal{Q}}\mathcal{N}$). For the Hecke category we follow the exposition given in [HW18, §2.5-2.7].

3.1. Realizations. Recall that a realization, as defined in [EW16, §3.1] consists of a commutative ring \mathbb{k} and a free and finitely generated \mathbb{k} -module \mathfrak{h} together with subsets

$$\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$$

of “roots” and “coroots” such that $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ for all $s \in S$, such that the formulas

$$s(v) := v - \langle v, \alpha_s \rangle \alpha_s^\vee \quad \text{for } s \in S \text{ and } v \in \mathfrak{h},$$

define an action of W on \mathfrak{h} and such that a technical condition on 2-colored quantum numbers (condition (3.3) in [EW16, §3.1]) is satisfied.

Unless otherwise stated we will assume in this paper that $\mathbb{k} = \mathbb{R}$ and that \mathfrak{h} is the “dual geometric representation” of W , i.e. we first choose a vector space \mathfrak{h} with $\mathfrak{h}^* = \bigoplus_{s \in S} \mathbb{R} \alpha_s$, and then define the elements $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ by the equations

$$(3.1) \quad \langle \alpha_t^\vee, \alpha_s \rangle = -2 \cos(\pi/m_{st})$$

(by convention $m_{ss} = 1$ and $\pi/\infty = 0$). Note that the subset $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$ is linearly independent if and only if W is finite (see the Theorem in §6.4 of [Hum90]). One can prove that the technical condition on 2-colored quantum numbers mentioned above is satisfied in this case using the analogue result for the geometric representation, because the quantum numbers of both realizations agree.

Remark 3.1. It is very important in this paper that the simple roots $\{\alpha_s\} \subset \mathfrak{h}^*$ are linearly independent. One can consider other realizations but they may introduce extra subtleties. For example one may not have the Parabolic Property. That is why we make the assumptions above.

Let $R = S(\mathfrak{h}^*)$ be the ring of regular functions on \mathfrak{h} or, equivalently, the symmetric algebra of \mathfrak{h}^* over \mathbb{k} . We see R as a graded \mathbb{k} -algebra by declaring $\deg \mathfrak{h}^* = 2$. The action of W on \mathfrak{h}^* extends to R by functoriality. For any $s \in S$, let $\partial_s : R \rightarrow R[-2]$ be the *Demazure operator* defined by the formula

$$\partial_s(f) = \frac{f - sf}{\alpha_s}.$$

In [EW16, §3.3] it is proved that this is well defined under our assumptions.

3.2. Towards the morphisms in \mathcal{H}_{BS} . An S -graph is a finite, planar, decorated graph with boundary properly embedded in the planar strip $\mathbb{R} \times [0, 1]$. Its edges are colored by S . The vertices in this graph are of 3 types:

(1) univalent vertices (“dots”): 

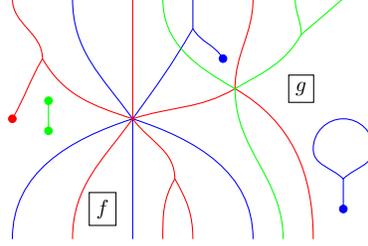
(2) trivalent vertices: 

(3) $2m_{rb}$ -valent vertices: 

We require that there are exactly $2m_{rb} < \infty$ edges originating from the vertex. They alternate in color between two different elements $r, b \in S$ around the vertex. The pictured example has $m_{rb} = 8$.

Additionally any S -graph may have its regions (the connected components of the complement of the graph in $\mathbb{R} \times [0, 1]$) decorated by boxes containing homogenous elements of R .

The following is an example of an S -graph with $m_{b,r} = 5$, $m_{b,g} = 2$, $m_{g,r} = 3$:



where f and g are homogeneous polynomials in R .

The *degree* of an S -graph is the sum over the degrees of its vertices and boxes. Each box has degree equal to the degree of the corresponding element of R . The vertices have degrees given by the following rule: dots have degree 1, trivalent vertices have degree -1 and $2m$ -valent vertices have degree 0. For example, the degree of the S -graph above is

$$+5 - 5 + \deg f + \deg g = \deg f + \deg g.$$

The intersection of an S -graph with $\mathbb{R} \times \{0\}$ (resp. with $\mathbb{R} \times \{1\}$) is a sequence of colored points called *bottom boundary* (resp. *top boundary*). In our example, the bottom (resp. top) boundary of the S -graph is (b, r, b, r, r, b, g, r) (resp. (r, b, r, g, b, r, g, g)).

3.3. Relations in \mathcal{H}_{BS} . Let us define the Hecke category. In this section we will give a summary of the central result of [EW16].

We define \mathcal{H}_{BS} as the monoidal category with objects sequences \underline{w} in S . If \underline{x} and \underline{y} are two such sequences, we define $\text{Hom}_{\mathcal{H}_{BS}}(\underline{x}, \underline{y})$ as the free R -module generated by isotopy classes of S -graphs with bottom boundary \underline{x} and top boundary \underline{y} , modulo the local relations below. Hom spaces are graded by the degree of the graphs (all the relations below are homogeneous). The structure of this monoidal category is given by horizontal concatenation of diagrams for the tensor product of morphisms and vertical concatenation of diagrams for the composition of morphisms.

In what follows, the rank of a relation is the number of colors involved in the relation. We use the color red for r and blue for b .

3.3.1. Rank 1 relations. Frobenius unit:

$$(3.2) \quad \text{Diagram 1} = \text{Diagram 2} .$$

Frobenius associativity:

$$(3.3) \quad \text{Diagram 1} = \text{Diagram 2} .$$

Needle relation:

$$(3.4) \quad \text{Diagram 1} = 0 .$$

Barbell relation:

$$(3.5) \quad \text{Diagram 1} = \text{Diagram 2} .$$

Nil Hecke relation:

$$(3.6) \quad \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} .$$

(See §3.1 for the definition of ∂_r .)

3.3.2. Rank 2 relations. Two-color associativity: We give the first three cases i.e. $m_{rb} = 2, 3, 4$. It is not hard to guess this relation for arbitrary m_{rb} (see [Eli16, 6.12] for details).

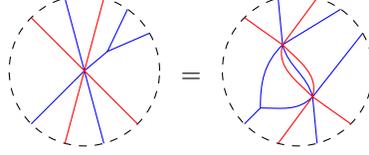
$m_{rb} = 2$ (type $A_1 \times A_1$):

$$\text{Diagram 1} = \text{Diagram 2} .$$

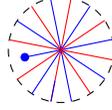
$m_{rb} = 3$ (type A_2):

$$\text{Diagram 1} = \text{Diagram 2} .$$

$m_{rb} = 4$ (type B_2):

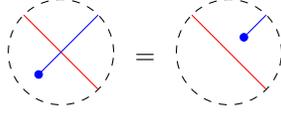


Elias' Jones-Wenzl relation: This relation expresses a dotted $2m_{rb}$ -vertex

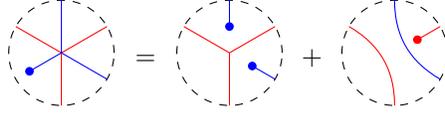


as a linear combination over R of diagrams consisting only of trivalent vertices and dots (no $2m_{rb}$ -valent vertices). We present again the first three cases i.e. $m_{rb} = 2, 3, 4$ (this time it is not easy to guess the general form, see [Eli16, 6.13] for all the details).

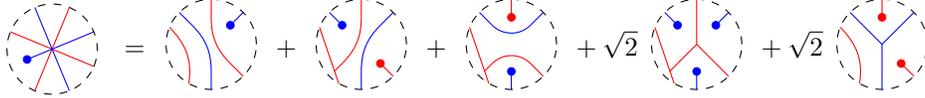
$m_{rb} = 2$ (type $A_1 \times A_1$):



$m_{rb} = 3$ (type A_2):



$m_{rb} = 4$ (type B_2):



3.3.3. *Rank 3 relations.* We will not repeat the definition of the Zamolodchikov relations here, and instead refer the reader to [EW16, §1.4.3]. This concludes the definition of \mathcal{H}_{BS} .

3.4. **The categories \mathcal{H} and \mathcal{H}_I .** If $M = \bigoplus_i M^i$ is a \mathbb{Z} -graded object, we denote by $M(1)$ its grading shift, i.e. $M(1)^i = M^{i+1}$. If $p = \sum_j a_j v^j \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$, we denote

$$p \cdot M = \bigoplus_j M(j)^{\oplus a_j}.$$

Given an additive category \mathcal{A} we denote by $[\mathcal{A}]$ its split Grothendieck group. If in addition \mathcal{A} has homomorphism spaces enriched in graded vector spaces we denote by \mathcal{A}^\oplus its additive graded envelope. That is, objects are formal finite direct sums $\bigoplus a_i(m_i)$ for certain objects $a_i \in \mathcal{A}$ and “grading shifts” $m_i \in \mathbb{Z}$. Homomorphism spaces in \mathcal{A}^\oplus are given by

$$\text{Hom}_{\mathcal{A}^\oplus}(\bigoplus a_i(m_i), \bigoplus a'_j(m'_j)) := \bigoplus_j \text{Hom}(a_i, a'_j)(m'_j - m_i).$$

We denote by $\mathcal{A}^{\oplus,0}$ the category with the same objects as \mathcal{A}^{\oplus} but with homomorphism spaces given by the degree zero morphisms in \mathcal{A}^{\oplus} :

$$\mathrm{Hom}_{\mathcal{A}^{\oplus,0}}(b, b) := \mathrm{Hom}_{\mathcal{A}^{\oplus}}(b, b)^0.$$

Both \mathcal{A}^{\oplus} and $\mathcal{A}^{\oplus,0}$ are equipped with a grading shift functor $b \mapsto b(1)$ given on objects by $\bigoplus a_i(m_i) \mapsto \bigoplus a_i(m_i + 1)$. Of course \mathcal{A}^{\oplus} is recoverable from $\mathcal{A}^{\oplus,0}$ and the grading shift functor (1). Finally we define \mathcal{A}^e to be the Karoubian envelope of $\mathcal{A}^{\oplus,0}$. In this setting, given objects $b, b' \in \mathcal{A}^e$ we abbreviate:

$$\begin{aligned} \mathrm{Hom}(b, b') &:= \mathrm{Hom}_{\mathcal{A}^e}(b, b'), \\ \mathrm{Hom}^\bullet(b, b') &:= \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(b, b'(m)). \end{aligned}$$

By definition the Hecke category \mathcal{H} is $\mathcal{H}_{\mathrm{BS}}^e$. If $I \subset S$, we define $\mathcal{H}_I := \mathcal{H}_{\mathrm{BS},I}^e$ where $\mathcal{H}_{\mathrm{BS},I}$ is the diagrammatic category obtained by replacing S by I in the definition. (That is, all diagrams in $\mathcal{H}_{\mathrm{BS},I}$ are only allowed to be colored by elements of I and decorated by the symmetric algebra of $\bigoplus_{s \in I} \mathbb{R}\alpha_s$ over \mathbb{R} .)

3.5. Basic facts about \mathcal{H} . Let us recall some terminology and notations from [EW16]. A *subsequence* of an expression $\underline{x} = s_1 s_2 \dots s_m$ is a sequence $\pi_1 \pi_2 \dots \pi_m$ such that $\pi_i \in \{e, s_i\}$ for all $1 \leq i \leq m$. Instead of working with subsequences, we work with the equivalent datum of a sequence $\mathbf{e} = \mathbf{e}_1 \mathbf{e}_2 \dots \mathbf{e}_m$ of 1's and 0's giving the indicator function of a subsequence, which we refer to as a *01-sequence*.

The *Bruhat stroll* is the sequence $x_0 = e, x_1, \dots, x_m$ defined by

$$x_i := s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2} \dots s_i^{\mathbf{e}_i}$$

for $0 \leq i \leq m$. We call x_i the *i^{th} -point* and x_m the *end-point* of the Bruhat stroll. We denote x_m by $\underline{x}^{\mathbf{e}}$. Alternatively, we will say that a subsequence \mathbf{e} of \underline{x} *expresses* the end-point $\underline{x}^{\mathbf{e}}$.

Let \mathbf{e} and \mathbf{f} be two 01-sequences of $\underline{x} = s_1 s_2 \dots s_m$ and let their corresponding Bruhat strolls be x_0, x_1, \dots, x_m and y_0, y_1, \dots, y_m . We say that $\mathbf{e} \geq \mathbf{f}$ in the *path dominance order* if $x_i \geq y_i$ for all $0 \leq i \leq m$. We define the *double path dominance order* (a partial order) on pairs (\mathbf{e}, \mathbf{f}) , where $(\mathbf{e}_1, \mathbf{f}_1) \leq (\mathbf{e}_2, \mathbf{f}_2)$ if $\mathbf{e}_1 \leq \mathbf{e}_2$ and $\mathbf{f}_1 \leq \mathbf{f}_2$.

Light leaves and *Double leaves* for Soergel bimodules were introduced in [Lib08] and [Lib15]. They give bases, as R -modules of the Hom spaces between Bott-Samelson bimodules. We recommend reading the paper [Lib15] in order to get used to these combinatorial objects and to read §6.1–6.3 of [EW16], where these bases are explained diagrammatically.

In [EW16, Definition 6.24] the authors define a character map $\mathrm{ch} : [\mathcal{H}] \rightarrow H$ and in [EW16, Corollary 6.27] they prove that it is an isomorphism. This is the reason why we call \mathcal{H} the Hecke category.

Following Soergel's classification of indecomposable Soergel bimodules, in [EW16, Theorem 6.26] the authors prove that the indecomposable objects in \mathcal{H} are indexed by W modulo shift, and they call B_w the indecomposable object corresponding to $w \in W$. It happens that the object B_s is the sequence with one element $(s) \in \mathcal{H}$. Because of this, if $\underline{w} = (s, r, \dots, t)$ we will sometimes denote by $B_{\underline{w}} := B_s B_r \dots B_t$ the element $\underline{w} \in \mathcal{H}$.

Let us refer to the reflection faithful representation of W over \mathbb{R} that Soergel constructs [Soe00, §2] as the *Kac-Moody representation* V_{KM} . The representation

V_{KM} is self-dual. By definition of V_{KM} one has that $\mathfrak{h}^* \subset V_{\text{KM}}$. Using the mentioned self-duality, we obtain an injection of W -representations $i : \mathfrak{h}^* \hookrightarrow V_{\text{KM}}^*$. This extends to an injection of symmetric algebras $R = S(\mathfrak{h}^*) \hookrightarrow S(V_{\text{KM}}^*)$. This means that one can see the diagrammatic Hecke category \mathcal{H} as a subcategory of the diagrammatic Hecke category associated with the Kac-Moody representation, that we denote $\mathcal{H}(V_{\text{KM}})$. The latter category is equivalent to the category of Soergel bimodules $\mathcal{B}(V_{\text{KM}})$ as proved in [EW16]. This, and the main result of [EW14] imply that $\text{ch}([B_w]) = b_w$. Thus the indecomposable objects in \mathcal{H} categorify the Kazhdan-Lusztig basis.

3.6. The anti-spherical category \mathcal{N} . Fix a subset $I \subset S$. We define the *Bott-Samelson anti-spherical category* \mathcal{N}_{BS} to be the category \mathcal{H}_{BS} quotiented by the ideal of all objects indexed by I -sequences. The *anti-spherical category* \mathcal{N} is the graded additive Karoubian completion of \mathcal{N}_{BS} , i.e. $\mathcal{N} := \mathcal{N}_{\text{BS}}^e$. For $x \in W$ we call D_x the image of B_x in the anti-spherical category \mathcal{N} and for y and expression, we call N_y the image in \mathcal{N} of the object B_y , or in other words, $N_y = N_{\text{id}} \cdot B_y$, where N_{id} is the image of the empty sequence.

We define the category \mathcal{N}' to be the category \mathcal{H} quotiented by the ideal of all objects $B_x \in \mathcal{H}$, with $x \notin W$.

Proposition 3.2. *There is an equivalence of categories $\mathcal{N} \cong \mathcal{N}'$.*

Proof. Consider the monoidal functor $\mathcal{F}_1 : \mathcal{H}_{\text{BS}}^{\oplus,0} \rightarrow \mathcal{N}'$ defined as the composition of the inclusion functor $\mathcal{H}_{\text{BS}}^{\oplus,0} \hookrightarrow \mathcal{H}$ with the canonical projection $\mathcal{H} \rightarrow \mathcal{N}'$.

Let $s \in S$ and $x \in W$ be such that $xs > x$. If $b_s b_x = b_{sx} + \sum_{y < sx} m_y b_y$, we have that $m_y \in \mathbb{Z}_{\geq 0}$ and that $m_y \neq 0 \Rightarrow sy < y$. Let \underline{w} be an I -sequence, say $\underline{w} = (s, s_1 \dots, s_n)$ with $s \in I$. Consider the decomposition of the sequence $(s_1 \dots, s_n)$ into indecomposable summands $\oplus_z p_z \cdot B_z$, with $p_z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. This gives a decomposition $\underline{w} = \oplus_z p_z \cdot B_s B_z$. This we can rewrite as $\underline{w} = \oplus_u p'_u \cdot B_u$, with $p'_u \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. Every B_u appearing in a non-zero term of this sum is such that $su < u$, thus $u \notin W$, and by definition they are zero in \mathcal{N}' . So the functor \mathcal{F}_1 factors through the ideal generated by all I -sequences, giving a functor $\mathcal{F}_2 : \mathcal{N}_{\text{BS}}^{\oplus,0} \rightarrow \mathcal{N}'$. The category \mathcal{N}' is idempotent complete, so the functor \mathcal{F}_2 lifts to a functor between the corresponding Karoubian completions $\mathcal{F}_3 : \mathcal{N} \rightarrow \mathcal{N}'$.

We will now prove that \mathcal{F}_3 is an equivalence of categories by finding an inverse equivalence $\mathcal{G}_3 : \mathcal{N}' \rightarrow \mathcal{N}$. Let $\mathcal{G}_1 : \mathcal{H}_{\text{BS}}^{\oplus,0} \rightarrow \mathcal{N}_{\text{BS}}^{\oplus,0}$ be the lift to the graded envelope of the canonical projection $\mathcal{H}_{\text{BS}} \rightarrow \mathcal{N}_{\text{BS}}$. The functor \mathcal{G}_1 lifts to a functor between the corresponding Karoubian completions $\mathcal{G}_2 : \mathcal{H} \rightarrow \mathcal{N}$. This functor is zero on any $B_x \in \mathcal{H}$ such that $x \notin W$ because any such element is a summand of an I -sequence. This gives us a functor $\mathcal{G}_3 : \mathcal{N}' \rightarrow \mathcal{N}$ that is clearly an inverse equivalence to \mathcal{F}_3 . \square

3.7. $Q\mathcal{N}$: a localization of \mathcal{N} . We will see in this section that a certain localized version of \mathcal{N} is very simple. Thus the situation for \mathcal{N} is similar (in terms of simplicity) to that of \mathcal{H} (see [EW16]). This result was unexpected (at least to the authors).

For $I \subseteq S$, define the ring $R_I := R/\langle \alpha_s \mid s \in I \rangle$. It is the largest quotient on which the parabolic group W_I acts trivially. If A is either the ring R or the ring R_I , we use the notation $A(\frac{1}{\Phi_I})$ for the localization of A by all the roots $\alpha \in \Phi$ that are not in Φ_I . In formulas, $A(\frac{1}{\Phi_I}) = A[\alpha^{-1} \mid \alpha \in \Phi \text{ and } \alpha \notin \Phi_I]$. We define

$Q_I := R_I(\frac{1}{\Phi_I^c})$ (i.e. “kill I and invert the rest”). Define the category

$${}_{Q_I}\mathcal{N}_{\text{BS}} := Q_I \otimes_{R_I} \mathcal{N}_{\text{BS}}.$$

This tensor product notation means that the objects of ${}_{Q_I}\mathcal{N}_{\text{BS}}$ are the same as the objects of \mathcal{N}_{BS} and $\text{Hom}_{{}_{Q_I}\mathcal{N}_{\text{BS}}}(X, Y) := Q_I \otimes_{R_I} \text{Hom}_{\mathcal{N}_{\text{BS}}}(X, Y)$. We remark that if $s \in I$ then α_s is zero in \mathcal{N}_{BS} (because of the Barbell relation). That is why R_I acts on the left of $\text{Hom}_{\mathcal{N}_{\text{BS}}}(X, Y)$. Another remark is that Q_I is ungraded, and so it is the category ${}_{Q_I}\mathcal{N}_{\text{BS}}$. Finally, we define the object of study of the following section ${}_{Q_I}\mathcal{N} := ({}_{Q_I}\mathcal{N}_{\text{BS}})^e$.

The right action of \mathcal{H}_{BS} on \mathcal{N}_{BS} extends in the obvious way to a right action of \mathcal{H}_{BS} on ${}_{Q_I}\mathcal{N}_{\text{BS}}$ (it is easy to check that this is indeed an action, i.e. to check the coherence conditions). Then, if a monoidal category acts on some category, its idempotent completion acts on the idempotent completion of the category. Thus, the category ${}_{Q_I}\mathcal{N}$ is a right \mathcal{H} -module.

Notation 3.3. When the context is clear, we will denote the identity morphism $\text{id}_M : M \rightarrow M$, just by M .

The following theorem will be proved in the next section.

Theorem 3.4. *In ${}_{Q_I}\mathcal{N}$ there is a set of objects $\{K_x\}_{x \in {}^I W}$ satisfying the following properties.*

- (1) $K_{\text{id}} = N_{\text{id}}$ (the image in ${}_{Q_I}\mathcal{N}$ of the empty sequence in \mathcal{H}_{BS}).
- (2) $K_x f = x(f)K_x$ for $f \in R$.
- (3) For all $x \in {}^I W$ we have $K_x B_s \cong \begin{cases} K_x \oplus K_{xs} & \text{if } xs \in {}^I W, \\ 0 & \text{if } xs \notin {}^I W. \end{cases}$
- (4) For all $x, y \in {}^I W$ we have $\text{Hom}(K_x, K_y) = \delta_{x,y} Q_I \cdot \text{id}_{K_x}$ (where $\delta_{x,y}$ is the Kronecker delta).
- (5) Any object in ${}_{Q_I}\mathcal{N}$ is isomorphic to a direct sum of K_x 's.

Remark 3.5. In particular, part (4) of this Theorem implies that $0 \neq K_x \in {}_{Q_I}\mathcal{N}$.

Remark 3.6. For a subset $J \subset S$ one can consider the localisation $R[J^{-1}] := R[\alpha^{-1}]_{\alpha \in \Phi_J}$ and the corresponding category of localised diagrammatic Soergel bi-modules $\mathcal{H}[J^{-1}] := R[J^{-1}] \otimes_R \mathcal{H}$. (The case when R is the fraction field is discussed in [EW16, §1.6]. Partial localisations also make sense.) A fundamental aspect of the current article is that $\mathcal{H}[J^{-1}]$ usually does *not* act on ${}_{Q_I}\mathcal{N}$. For example, if $s \in I \cap J \neq \emptyset$ then α_s is an invertible morphism in $\mathcal{H}[J^{-1}]$ but acts as zero on $K_{\text{id}} \in {}_{Q_I}\mathcal{N}$. (A more mundane way of seeing that $\mathcal{H}[S^{-1}]$ cannot act on ${}_{Q_I}\mathcal{N}$ follows by observing that some of the structure constants in the standard bases for the action of the Coxeter group on the anti-spherical module are negative.)

However, there is one situation where part of the localised category does act. Suppose that $x \in {}^I W$ and $J \subset S$ satisfies $xW_J \subset {}^I W$. One can deduce, using the Parabolic Property, that $x(\alpha) \notin \Phi_I$ for all $\alpha \in \Phi_J$. Thus right action by α is invertible on $K_x \in {}_{Q_I}\mathcal{N}$ for all $\alpha \in \Phi_J$. By the universal property of localization³, $\mathcal{H}[J^{-1}]$ acts on the full subcategory generated by K_x . In this case, for any $u \in W_J$ one has a canonical isomorphism $K_x \cdot Q_u = K_{xu}$, where $Q_u \in \mathcal{H}[J^{-1}]$ denotes the object considered in [EW16, §5.4].

³Recall that another definition of an action $\star : \mathcal{M} \times A \rightarrow A$ of a monoidal category \mathcal{M} on a category A is a strong monoidal functor $F : \mathcal{M} \rightarrow \text{End}(A)$ into the monoidal category of endofunctors of A .

4. PROOF OF THEOREM 3.4

Proposition 4.1. $\text{End}_{\mathcal{N}}(D_{\text{id}}) = R_I$.

Proof. By definition of the category \mathcal{N} , we have

$$\text{End}_{\mathcal{N}}(D_{\text{id}}) = \text{End}_{\mathcal{H}}(B_{\text{id}})/J,$$

where J is the ideal of $\text{End}_{\mathcal{H}}(B_{\text{id}})$ generated as an R -module by maps that factor through an I -sequence.

By the double leaves theorem in \mathcal{H} we know that $\text{End}_{\mathcal{H}}(B_{\text{id}}) = R$ (the only double leaf in $\text{End}_{\mathcal{H}}(B_{\text{id}})$ is the identity). So, if one defines the ideal

$$\alpha_I := \langle \alpha_s : s \in I \rangle \subset R,$$

to finish the proof we just need to prove that $J = \alpha_I$.

It is easy to see that $J \supset \alpha_I$, because if $s \in I$, $\alpha_s \in \text{End}_{\mathcal{H}}(R)$ can be factored through B_s :

$$R \xrightarrow{\bullet} B_s \xrightarrow{\bullet} R$$

Let us prove that $J \subset \alpha_I$. Any map in $f \in J$ can be written as

$$f = \sum_{\substack{\underline{x} \text{ is an} \\ I\text{-sequence}}} p_{\underline{x}}(g_{\underline{x}} \circ h_{\underline{x}}),$$

where $p_{\underline{x}}$ is an element of R , $g_{\underline{x}} : \underline{x} \rightarrow B_{\text{id}}$ and $h_{\underline{x}} : B_{\text{id}} \rightarrow \underline{x}$. By the double leaves theorem, each $g_{\underline{x}}$ can be written as an R -linear combination of light leaves. We remark that we don't mean double leaves but honest light leaves, given that if the codomain is B_{id} , double leaves are light leaves. The same can be said of $h_{\underline{x}}$ (with upside-down light leaves).

Thus it is enough to prove that if $f = g_{\underline{x}} \circ h_{\underline{x}}$, with \underline{x} an I -sequence, $g_{\underline{x}}$ a light leaf and $h_{\underline{x}}$ an upside-down light leaf, then $f \in \alpha_I$. We prove this by induction on the length of \underline{x} .

Suppose that \underline{x} has s in its left-most position. If $g_{\underline{x}}$ and $h_{\underline{x}}$ have an $U0$ in the left-most position then $f \in \langle \alpha_s \rangle$ and we are done.

Suppose that either $g_{\underline{x}}$ or $h_{\underline{x}}$ have an $U0$ that is not in the left-most position. To fix ideas, say that it is the case for $g_{\underline{x}}$. Then one can write $g_{\underline{x}}$ as the composition of a dot and $g_{\underline{y}} : \underline{y} \rightarrow B_{\text{id}}$, with \underline{y} and I -sequence and $l(\underline{y}) < l(\underline{x})$ as in the picture:

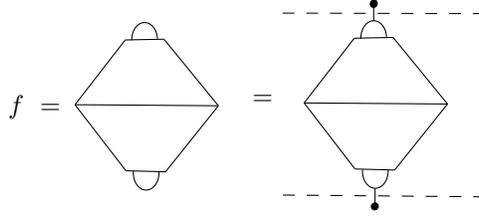
$$f = \left. \begin{array}{c} \triangle \\ \hline g_{\underline{y}} \\ \hline \bullet \\ \hline h_{\underline{x}} \\ \triangle \end{array} \right\} g_{\underline{x}}$$

By the double leaves theorem, the part of the diagram that is below $g_{\underline{y}}$ (i.e. the composition of $h_{\underline{x}}$ and the dot) can be written as an R -linear combination of leaves $\underline{y} \rightarrow B_{\text{id}}$ flipped upside-down, so by the induction hypothesis we are done.

So we are left with two cases: the first case is that the light leaves of f (i.e. $g_{\underline{x}}$ and $h_{\underline{x}}$) don't have $U0$'s. The second case is that one of them has one $U0$ in the left-most position and the other one has no $U0$'s.

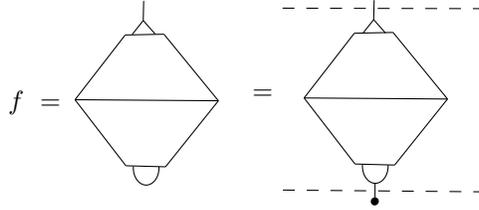
As the degree of a light leaf is the number of its $U0$'s minus the number of its $D1$'s, and there are no negative degree maps in $\text{End}(B_{\text{id}})$, we conclude that in the first case both light leaves of f have no $D1$'s and in the second case there is at most one $D1$ occurring in both light leaves.

In the first case, the last step of both light leaves of f is a $D0$. Thus we have:



The second equality is by the definition of the cup and cap. The map between the dotted lines is a negative degree map (degree -2) in $\text{Hom}(B_s, B_t)$, where s can be the same as t . This implies that $f = 0$.

If we are in the second case then either we can conclude as in the previous paragraph, or the last step of one of the light leaves of f is a $D1$ (say, of g_x):



Again, the map between the dotted lines is a negative degree map (degree -1) in $\text{Hom}(B_s, B_t)$. This implies that $f = 0$ and we conclude the proof. \square

Corollary 4.2. $\text{End}_{Q\mathcal{N}}(B_{\text{id}}) = Q_I$.

We now turn to the proof of Theorem 3.4. It relies on a modest amount of homological algebra. Given an additive category, \mathcal{A} we denote by $K^b(\mathcal{A})$ the homotopy category of bounded complexes in \mathcal{A} . It is a triangulated category. If \mathcal{A} is in addition monoidal, then $K^b(\mathcal{A})$ is monoidal under tensor product of complexes. If \mathcal{M} is a right \mathcal{A} -module, then $K^b(\mathcal{M})$ is a right $K^b(\mathcal{A})$ -module.

In particular, $K^b(\mathcal{H})$ is a monoidal category, and it has right modules $K^b(\mathcal{N})$ and $K^b({}_Q\mathcal{N})$. An important role will be played by Rouquier complexes. For any $s \in S$ consider the complex

$$F_s := 0 \rightarrow B_s \xrightarrow{\bullet} R(1) \rightarrow 0$$

where B_s is in degree 0. It is known that F_s is an invertible element of $K^b(\mathcal{H})$, with inverse

$$F_s^{-1} := 0 \rightarrow R(-1) \xrightarrow{\bullet} B_s \rightarrow 0$$

where B_s is again in degree zero. Moreover, given any element $w \in W$ we set

$$F_w = F_{\underline{w}} := F_s F_t \cdots F_u,$$

where $w := st \cdots u$ is a reduced expression for w . This complex does not depend on the reduced expression chosen. The above results are due to Rouquier [Rou06]⁴. The reader may consult [AMRW19] for an in-depth discussion of Rouquier complexes in the diagrammatic language.

The following beautiful little lemma is apparently well-known in the link homology literature (see e.g. [GH17]):

Lemma 4.3. *We have $F_x \cdot f = x(f) \cdot F_x$ as endomorphisms of $F_x \in K^b(\mathcal{H})$.*

Proof. It is enough to check this for $x = s \in S$ a simple reflection and f a homogeneous polynomial. In this case we need to check that $s(f) \cdot F_s - F_s \cdot f$ is null-homotopic. This map of complexes is

$$\begin{array}{ccc} B_s & \longrightarrow & R(1) \\ a_0 \downarrow & & \downarrow a_1 \\ B_s(d) & \longrightarrow & R(d+1) \end{array}$$

where $d = \deg f$ and

$$a_0 = \boxed{s(f)} - \boxed{f} = \boxed{-\partial_s f}$$

and

$$a_1 = \boxed{s(f) - f}.$$

Now one checks directly that

$$h = \boxed{-\partial_s f} : R(1) \rightarrow B_s(d).$$

provides the null-homotopy. \square

We now turn to the proof in earnest. Define K_{id} to be the image in ${}_{\mathcal{Q}}\mathcal{N}$ of D_{id} , also known as the empty sequence in \mathcal{H}_{BS} .

Proposition 4.4. *If $x \in {}^I W$ then $K_{\text{id}} \cdot F_x$ is isomorphic to a complex concentrated in degree zero. Moreover, we have an isomorphism*

$$K_{\text{id}} \cdot F_x \cong K_{\text{id}} \cdot (F_{x^{-1}})^{-1}$$

In other words, once we have proved the proposition we know that there exist objects

$$K_x \in {}_{\mathcal{Q}}\mathcal{N} \quad \text{for each } x \in {}^I W$$

such that

$$K_x \cong K_{\text{id}} \cdot F_x \in K^b({}_{\mathcal{Q}}\mathcal{N}).$$

These objects will play a key role in the proof of Theorem 3.4.

⁴In fact, [Rou06] shows that F_w is defined up to canonical isomorphism. We won't need this stronger statement below.

Proof. We will prove the proposition by induction on the length of x , with both statements in case $\ell(x) = 0$ being trivial. Write $x = ys$ with $\ell(x) = \ell(y) + 1$ and $y \in {}^I W$. By induction, there exists $K_y \in {}_Q \mathcal{N}$ such that

$$K_y \cong K_{\text{id}} \cdot F_y \cong K_{\text{id}} \cdot (F_{y^{-1}})^{-1} \in K^b({}_Q \mathcal{N}).$$

In particular, $K_{\text{id}} \cdot F_x = K_{\text{id}} \cdot F_y F_s$ is isomorphic to the two-term complex

$$(4.1) \quad \dots \rightarrow 0 \rightarrow K_y B_s \xrightarrow{K_y \blacktriangleright} K_y R \rightarrow \dots$$

whilst $K_{\text{id}} \cdot (F_{x^{-1}})^{-1} = K_{\text{id}} \cdot (F_{y^{-1}})^{-1} F_s^{-1}$ is isomorphic to the two-term complex

$$(4.2) \quad \dots \rightarrow K_y R \xrightarrow{K_y \blacktriangleleft} K_y B_s \rightarrow 0 \rightarrow \dots$$

with $K_y B_s$ in degree zero in both complexes. The composition

$$(4.3) \quad K_y R \xrightarrow{K_y \blacktriangleleft} K_y B_s \xrightarrow{K_y \blacktriangleright} K_y R$$

is equal to $K_y \alpha_s$. By Lemma 4.3, we have

$$K_y \alpha_s = K_{\text{id}} \cdot F_y \alpha_s = y(\alpha_s) \cdot F_y$$

which is invertible by the parabolic property 2.3. In particular we can find an isomorphism

$$K_y B_s \cong K_y \oplus X$$

such that the differentials in (4.1) (resp. (4.2)) are (up to a scalar) the projection (resp. inclusion) of $K_y = K_y R$. Removing this contractible summand, we deduce that

$$X \cong K_{\text{id}} \cdot F_x \cong K_{\text{id}} \cdot (F_{x^{-1}})^{-1} \in K^b({}_Q \mathcal{N}).$$

and the proposition follows. \square

The following details the behaviour of K_x under F_s in general.

Proposition 4.5. *For any $x \in {}^I W$ we have*

$$K_x \cdot F_s \cong \begin{cases} K_{xs} & \text{if } xs \in {}^I W, \\ K_x[-1] & \text{if } xs \notin {}^I W. \end{cases}$$

Remark 4.6. We leave it to the reader to formulate an analogous result for $(F_s)^{-1}$. In particular, the K_x are preserved under the action of the braid group. (We will not need this fact below).

Proof. Let us first assume $xs \in {}^I W$. The only case not already directly covered by Proposition 4.4 is when $xs < x$. But then

$$K_x \cdot F_s \cong K_{\text{id}} \cdot (F_{x^{-1}})^{-1} F_s \cong K_{\text{id}} \cdot (F_{(xs)^{-1}})^{-1} \cong K_{xs}.$$

by Proposition 4.4.

We now examine the second case. If $x \in {}^I W$ and $xs \notin {}^I W$ then $xs = tx$ for some $t \in I$ (proof in Section 2.3). We have

$$K_x F_s \cong K_{\text{id}} F_{xs} \cong K_{\text{id}} F_t F_x \stackrel{(*)}{\cong} K_{\text{id}}[-1] F_x \cong K_x[-1]$$

where for (*) we use that

$$D_{\text{id}}(B_t \rightarrow R) = (B_t \rightarrow R) = (0 \rightarrow R) = D_{\text{id}}[-1]$$

because $B_t = 0$ in \mathcal{N} . (We ignore internal internal shifts (i.e. (1)'s), as they don't affect the outcome.) \square

Proposition 4.7. *For $\text{id} \neq x \in {}^I W$ we have*

$$\text{Hom}_{\mathcal{Q}\mathcal{N}}(K_x, K_{\text{id}}) = 0.$$

Proof. Let us choose a reduced expression $\underline{x} = (s_1, \dots, s_m)$ for x . Consider the tensor product of complexes

$$F_{\underline{x}} = F_{s_1} F_{s_2} \dots F_{s_m} = (\dots \rightarrow 0 \rightarrow B_{\underline{x}} \xrightarrow{d_0} \bigoplus_{i=1}^m B_{x_i} \rightarrow \dots)$$

where:

- (1) $B_{\underline{x}}$ is in degree 0;
- (2) B_{x_i} indicates the tensor product of $B_{s_1} B_{s_2} \dots B_{s_m}$ with B_{s_i} omitted;
- (3) the differential d_0 is a direct sum of dot maps (up to sign).

We first claim that

$$\text{Hom}_{K^b(\mathcal{H})}(F_{\underline{x}}, R) = 0.$$

This is a special case of the main result in [LW14]. But this particular case is easy enough that can be proved directly. Unpacking the definitions, the equality holds if and only if

$$\text{Hom}_{\mathcal{H}}(\bigoplus_{i=1}^m B_{x_i}, R) \rightarrow \text{Hom}_{\mathcal{H}}(B_{\underline{x}}, R)$$

is surjective. However, this is the case by the light leaves theorem, because any map in $\text{Hom}_{\mathcal{H}}(B_{\underline{x}}, R)$ is an R -linear combination of light leaves, and each of these must have an $U0$ (i.e. a dot) after a certain number of $U1$'s, because $x \neq \text{id}$.

Now consider the commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{H}}(\bigoplus_{i=1}^m B_{x_i}, R) & \xrightarrow{(\mathcal{H})} & \text{Hom}_{\mathcal{H}}(B_{\underline{x}}, R) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{N}}(\bigoplus_{i=1}^m B_{x_i}, R) & \xrightarrow{(\mathcal{N})} & \text{Hom}_{\mathcal{N}}(B_{\underline{x}}, R) \\ \downarrow & & \downarrow \\ Q_I \otimes \text{Hom}_{\mathcal{N}}(\bigoplus_{i=1}^m B_{x_i}, R) & \xrightarrow{({}_{\mathcal{Q}}\mathcal{N})} & Q_I \otimes \text{Hom}_{\mathcal{N}}(B_{\underline{x}}, R) \end{array}$$

We have just argued that the arrow labelled (\mathcal{H}) is surjective, hence so is (\mathcal{N}) (the upper vertical maps are surjections by definition of the morphisms in a quotient category), and hence so is the arrow labelled $({}_{\mathcal{Q}}\mathcal{N})$ (as $Q_I \otimes (-)$ preserves surjections). This implies that

$$\text{Hom}_{\mathcal{Q}\mathcal{N}}(K_x, K_{\text{id}}) = \text{Hom}_{K^b({}_{\mathcal{Q}}\mathcal{N})}(K_x, K_{\text{id}}) = \text{Hom}_{K^b({}_{\mathcal{Q}}\mathcal{N})}(K_{\text{id}} F_{\underline{x}}, K_{\text{id}}) = 0$$

as claimed. \square

Remark 4.8. The objects $D_{\text{id}} \cdot F_x$ and $D_{\text{id}} \cdot F_{x^{-1}}$ in $K^b(\mathcal{N})$ (for $x \in {}^I W$) should satisfy vanishing conditions generalizing the well-known vanishing

$$\dim \text{Ext}^i(\Delta_{\lambda}, \nabla_{\mu}) = \delta_{0,i} \delta_{\lambda,\mu}$$

in highest weight categories. For the homotopy category of the Hecke category, this is proved in [LW14]. It is likely that the techniques of [Mak, AR16a, AR16b] are

most easily generalized to this setting. Such a vanishing result in $K^b(\mathcal{N})$ would imply most results in this section.

Proposition 4.9. *For $x, y \in {}^I W$ we have*

$$\mathrm{Hom}(K_x, K_y) = \begin{cases} Q_I & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Action by the equivalence F_x gives us identifications

$$Q_I = \mathrm{End}_{\mathcal{Q}\mathcal{N}}(K_{\mathrm{id}}) = \mathrm{End}_{K^b(\mathcal{Q}\mathcal{N})}(K_{\mathrm{id}}F_x) = \mathrm{End}_{\mathcal{Q}\mathcal{N}}(K_x)$$

from which the first statement follows.

We now proceed to the vanishing statement. By Proposition 4.4 we have

$$\mathrm{Hom}_{K^b(\mathcal{Q}\mathcal{N})}(K_x, K_y) = \mathrm{Hom}_{K^b(\mathcal{Q}\mathcal{N})}(K_x, K_{\mathrm{id}}(F_{y^{-1}})^{-1}) = \mathrm{Hom}_{K^b(\mathcal{Q}\mathcal{N})}(K_x F_{y^{-1}}, K_{\mathrm{id}})$$

By Proposition 4.5 we know that $K_x F_{y^{-1}}$ is isomorphic to $K_z[m]$ for some $z \in {}^I W$ and $m \leq 0$, with $z = \mathrm{id}$ and $m = 0$ if and only if $x = y$. If $m < 0$ we are done, as there are no maps between complexes concentrated in different degrees. If $m = 0$ then we are done by Proposition 4.7. \square

Finally we establish:

Proposition 4.10. *For $x \in {}^I W$ we have*

$$K_x B_s \cong \begin{cases} K_x \oplus K_{xs} & \text{if } xs \in {}^I W, \\ 0 & \text{if } xs \notin {}^I W. \end{cases}$$

Proof. The fact that $K_x B_s \cong K_x \oplus K_{xs}$ when $xs > x$ and $xs \in {}^I W$ follows from the proof of Proposition 4.4. We now consider the case when $xs < x$ (and then necessarily $xs \in {}^I W$). The stupid filtration on F_s^{-1} yields a distinguished triangle

$$B_s \rightarrow F_s^{-1} \rightarrow R(-1)[1] \xrightarrow{[1]}$$

which we may turn to yield

$$R(-1) \rightarrow B_s \rightarrow F_s^{-1} \xrightarrow{[1]}.$$

If we act on K_x with this triangle we obtain a distinguished triangle

$$K_x \rightarrow K_x B_s \rightarrow K_{xs} \xrightarrow{[1]}$$

and hence $K_x B_s \cong K_x \oplus K_{xs}$ because $\mathrm{Hom}(K_{xs}, K_x[1]) = 0$.

We now consider the case when $xs \notin {}^I W$. Then necessarily $xs = tx$ for some $t \in I$. We claim that in this case we have isomorphisms

$$(4.4) \quad F_x B_s \cong B_t F_s \quad \text{in } K^b(\mathcal{H}).$$

To establish (4.4), first note that $\mathrm{Hom}(R(1)[-1], F_s)$ (degree zero morphisms) is one-dimensional. Hence left and right action by the equivalences F_x allows us to deduce the same statement for $\mathrm{Hom}(F_x(1)[-1], F_x F_s)$ and $\mathrm{Hom}(F_x(1)[-1], F_t F_x)$.

In particular, we have a commutative diagram

$$(4.5) \quad \begin{array}{ccccccc} F_x(1)[-1] & \longrightarrow & F_x F_s & \longrightarrow & F_x B_s & \xrightarrow{[1]} & \longrightarrow \\ \downarrow \sim & & \downarrow \sim & & & & \\ F_x(1)[-1] & \longrightarrow & F_t F_x & \longrightarrow & B_t F_x & \xrightarrow{[1]} & \longrightarrow \end{array}$$

The triangles are obtained via action on the triangles

$$R(1)[-1] \rightarrow F_u \rightarrow B_u \xrightarrow{[1]}$$

by F_x on the left (with $u = s$) and right (with $u = t$). Now (4.4) follows by the existence of a (non-canonical) isomorphism of cones.

We are done:

$$K_x B_s = K_{\text{id}} \cdot F_x B_s \stackrel{(4.4)}{\cong} K_{\text{id}} \cdot B_t F_x = 0$$

because $B_t = 0$ in \mathcal{N} . \square

We now turn to the proof of the main theorem:

Proof of Theorem 3.4. (1) is immediate from the definitions, (2) follows from Lemma 4.3, (3) follows from Proposition 4.10, (4) follows from Proposition 4.9, and (5) is immediate from (3). \square

5. I -ANTISPHERICAL DOUBLE LEAVES ARE A BASIS

In this section we will follow the notation of [EW16, Construction 6.1], where light leaves and double leaves are explained in diagrammatic terms. However we make a slight modification of the construction therein. In the definition of ϕ_k start by doing the following. If $w_{k-1}s \notin {}^I W$ and e_k is either $U0$ or $U1$ then apply some loop in the rex graph starting (and ending) in $\underline{w}_{k-1}s$ and passing through an I -sequence. If not, do nothing. This slight modification in the construction changes nothing in the proof that these morphisms give bases of the corresponding Hom spaces.

Recall that if $LL_{\underline{x}, \mathbf{e}}: B_{\underline{x}} \rightarrow B_{\underline{w}}$ is a light leaves map where \underline{w} is a rex for w , by flipping this diagram upside-down, we get a map $\overline{LL}_{\underline{x}, \mathbf{e}}: B_{\underline{w}} \rightarrow B_{\underline{x}}$.

Let \underline{x} and \underline{y} be arbitrary sequences with subsequences \mathbf{e} and \mathbf{f} respectively, such that $(\underline{x}, \mathbf{e})$ and $(\underline{y}, \mathbf{f})$ both express w . Choose a rex \underline{w} for w , and construct maps $LL_{\underline{x}, \mathbf{e}}: B_{\underline{x}} \rightarrow B_{\underline{w}}$ and $\overline{LL}_{\underline{y}, \mathbf{f}}: B_{\underline{w}} \rightarrow B_{\underline{y}}$. The corresponding *double leaves map* is the composition

$$\mathbb{L}\mathbb{L}_{w, \mathbf{f}, \mathbf{e}} \stackrel{\text{def}}{=} \overline{LL}_{\underline{y}, \mathbf{f}} \circ LL_{\underline{x}, \mathbf{e}}.$$

Finally, $\mathbb{L}\mathbb{L}_{\underline{x}, \underline{y}}$ is the set consisting of all double leaves of the form $\mathbb{L}\mathbb{L}_{w, \mathbf{f}, \mathbf{e}}$ (for all $w \in W$ and all subexpressions \mathbf{e} and \mathbf{f} such that $(\underline{x}, \mathbf{e})$ and $(\underline{y}, \mathbf{f})$ both express w .)

Definition 5.1. A subexpression \mathbf{e} of $\underline{x} = s_1 s_2 \cdots s_m$ (we don't consider \underline{x} necessarily reduced) is *I -antispherical* if for all $0 \leq k < m$ we have

$$x_k s_{k+1} \in {}^I W,$$

where x_i is the i^{th} -point of the Bruhat stroll. An element of the set $LL_{\underline{x}, \mathbf{e}}$ is called an *I -antispherical light leaf* if \mathbf{e} is an I -antispherical subexpression of \underline{x} . An *I -antispherical double leaf* is a double leaf where both light leaves that compose it

are I -antispherical. Let \underline{y} be another expression. Then we define the set $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}^{Ias}$ as the subset of $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}$ consisting of I -antispherical double leaves. We will also denote by $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}^{Ias}$ this set in \mathcal{N} .

Remark 5.2. Let $J \subseteq I \subseteq S$. As ${}^I W \subseteq {}^J W$, we have that if a light leaf is I -antispherical, then it is also J -antispherical.

Theorem 5.3. *If $x, y \in {}^I W$, the set $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}^{Ias}$ forms a free R_I -basis for $\text{Hom}_{\mathcal{N}}^\bullet(N_{\underline{x}}, N_{\underline{y}})$ as a left module.*

Remark 5.4. If x or y are not in ${}^I W$, the theorem still holds, but it says that an empty set is a basis of the zero module.

Proof. In \mathcal{H} the set $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}$ generates over R (moreover is an R -basis for) the space $\text{Hom}_{\mathcal{H}}^\bullet(B_{\underline{x}}, B_{\underline{y}})$. By definition of \mathcal{N} we deduce that the set $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}$ generates over R the space $\text{Hom}_{\mathcal{N}}^\bullet(N_{\underline{x}}, N_{\underline{y}})$. As $\alpha_s = 0 \in \mathcal{N}$ if $s \in I$, we deduce that the set $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}$ generates over R_I the space $\text{Hom}_{\mathcal{N}}^\bullet(N_{\underline{x}}, N_{\underline{y}})$.

But it is easy to see that with the slightly modified construction of light leaves (discussed above) a light leaf that is not I -antispherical is zero in \mathcal{N} . Indeed, there is some $0 \leq k < m$ such that $x_k s_{k+1} \notin {}^I W$. Consider the lowest k with that property. If $k = 0$ then $s_1 \in I$ and the light leaf is zero. If $k > 1$ then $x_{k-1} s_k \in {}^I W$, thus in any case $x_k \in {}^I W$. By Fact (c) in Section §2.5, we have the inequality $x_k < x_k s_{k+1}$. So we have that e_{k+1} is either $U0$ or $U1$, so by the slight modification of the construction, the light leaf factors through some I -sequence, thus is zero. So we have proved that the I -antispherical double leaves generate the space $\text{Hom}_{\mathcal{N}}^\bullet(N_{\underline{x}}, N_{\underline{y}})$ over R_I .

The proof of the linear independence of light leaves and double leaves is very similar to the proof in the Hecke category, so we will be brief. First note that repeated application of the canonical decomposition (for $x \in {}^I W$)

$$K_x \cdot B_s = \begin{cases} K_x \oplus K_{xs} & \text{if } xs \notin {}^I W, \\ 0 & \text{if } xs \in {}^I W \end{cases}$$

of Theorem 3.4(3) gives a canonical decomposition

$$(5.1) \quad N_{\underline{x}} = \bigoplus_{\substack{\mathbf{e} \subseteq \underline{x} \\ I\text{-antispherical}}} K_{\mathbf{e}}.$$

Now consider an anti-spherical light leaf

$$LL_{\underline{x},\mathbf{e}} : N_{\underline{x}} \rightarrow N_{\underline{w}}$$

where \underline{w} is a reduced expression for $w \in {}^I W$. After localizing and projecting to the canonical summand $K_w \subset N_{\underline{w}}$ we get maps

$$p_{\mathbf{f}}^{\mathbf{e}} : K_{\mathbf{f}} \rightarrow K_w$$

for each I -antispherical subexpression $\mathbf{f} \subseteq \underline{x}$ for w . Because $Q_I = \text{Hom}(K_{\mathbf{f}}, K_w)$ (see Theorem 3.4(4)) we may regard $p_{\mathbf{f}}^{\mathbf{e}}$ as an element of Q_I .

Proposition 5.5. *We have that $p_{\mathbf{f}}^{\mathbf{e}} = 0$ unless $\mathbf{f} \leq \mathbf{e}$ in path dominance order. Moreover, $p_{\mathbf{e}}^{\mathbf{e}}$ is a non-zero product of the images of roots in Q_I , which is independent of the choice of light leaves.*

Proof. The proof is very similar to the proof of [EW16, Proposition 6.6]. (Note that the α_k which appears in the proof of [EW16, Proposition 6.6] has non-zero image in Q_I by the parabolic property.) \square

In ${}_Q\mathcal{N}$, the morphism $\mathbb{L}\mathbb{L}_{w,\mathbf{f},\mathbf{e}}$ gives a coefficient $p_{\mathbf{f}',\mathbf{e}'}^{\mathbf{f},\mathbf{e}} \in Q_I$ given by the inclusion of each standard summand $K_{\mathbf{e}'}$ of $N_{\underline{x}}$ and projection to each standard summand $K_{\mathbf{f}'}$ of $N_{\underline{y}}$, in the decomposition (5.1). The following facts about these coefficients are easy consequences of Proposition 5.5 (see also the discussion in [EW16, §6.3]):

- $p_{\mathbf{f}',\mathbf{e}'}^{\mathbf{f},\mathbf{e}} = 0$ unless $(\underline{x}, \mathbf{e}')$ and $(\underline{y}, \mathbf{f}')$ express the same element v . This is a direct consequence of Theorem 3.4(4).
- $p_{\mathbf{f}',\mathbf{e}'}^{\mathbf{f},\mathbf{e}} = 0$ unless both $\mathbf{e}' \leq \mathbf{e}$ and $\mathbf{f}' \leq \mathbf{f}$. This is a direct consequence of the construction of light leaves, the fact that the composition of the projection and the dot

$$B_{\text{id}} \xrightarrow{\bullet} B_s \longrightarrow K_s$$

and the composition of the dot and the inclusion

$$K_s \longrightarrow B_s \xrightarrow{\bullet} B_{\text{id}}$$

are both zero, and again Theorem 3.4(4).

- The element $p_{\mathbf{f}',\mathbf{e}'}^{\mathbf{f},\mathbf{e}}$ is invertible in Q_I . Moreover, it is a product of roots, obeying a simple formula independent of the choice of LL maps.

Consider the double path dominance order introduced in §3.5, restricted to pairs of 01-sequences with the same fixed end-point. As we have seen, $\mathbb{L}\mathbb{L}$ maps satisfy upper-triangularity with respect to this partial order, with an invertible diagonal, thus giving linear independence of $\mathbb{L}\mathbb{L}_{\underline{x},\underline{y}}^{I,\text{as}}$ over R_I . \square

6. CATEGORIFICATION THEOREM

Recall that in Section 3.6 we defined D_x as the image of the indecomposable object B_x in \mathcal{N} . By the definition of \mathcal{N} as a quotient of additive categories, it is clear that the set

$$\{D_x \mid x \in {}^I W\}$$

is a set of representatives for the isomorphism classes of indecomposable objects of \mathcal{N} , up to shift. Its image in ${}_Q\mathcal{N}$ is of the form

$$K_x \oplus \bigoplus_{y < x} m_y \cdot K_y$$

with $m_y \in \mathbb{N}$ (note that the Q_x are non-zero by Remark 3.5). For any $x \in {}^I W$ consider the full additive subcategory

$$\mathcal{N}_{\not\geq x} := \langle D_y(m) \mid y \not\geq x \text{ and } m \in \mathbb{Z} \rangle_{\oplus} \subseteq \mathcal{N},$$

and the quotient (of additive categories)

$$\mathcal{N}^{\geq x} := \mathcal{N} / \mathcal{N}_{\not\geq x}.$$

Lemma 6.1. *For any expression \underline{y} , $\text{Hom}_{\mathcal{N}^{\geq x}}(N_{\underline{y}}, D_x)$ is a free graded R_I -module, with basis the (images of) the I -antispherical light leaves corresponding to I -antispherical subexpressions of \underline{y} expressing x .*

Proof. Let \underline{x} be a reduced expression for x . By Theorem 5.3, $\text{Hom}_{\mathcal{N}}^{\bullet}(N_{\underline{y}}, N_{\underline{x}})$ is free over R_I with basis given by I -antispherical double leaves. However, when we pass to the quotient $\mathcal{N}^{\geq x}$, all double leaves with non-trivial upper light leaf factor through an object in $\mathcal{N}_{\not\geq x}$ and are therefore zero. We conclude that the claimed elements span $\text{Hom}_{\mathcal{N}^{\geq x}}^{\bullet}(N_{\underline{y}}, N_{\underline{x}})$.

To see that they are linearly independent, consider the chain of functors

$$\mathcal{N} \rightarrow {}_Q\mathcal{N} \rightarrow {}_Q\mathcal{N}/\langle K_z \mid z \not\geq x \rangle_{\oplus}$$

where the first functor is given by localisation, and the second is the quotient functor. If $y \not\geq x$, the image of D_y is zero, and hence we obtain a functor

$$\mathcal{N}^{\geq x} \rightarrow {}_Q\mathcal{N}/\langle K_z \mid z \not\geq x \rangle_{\oplus}.$$

By Proposition 5.5 the maps are linearly independent on the right hand side, and hence are on the left hand side too. Thus the statement of the lemma is true for $\text{Hom}_{\mathcal{N}^{\geq x}}^{\bullet}(N_{\underline{y}}, N_{\underline{x}})$. Finally, D_x and $N_{\underline{x}}$ are isomorphic in $\mathcal{N}^{\geq x}$ and the lemma follows. \square

Because any object in \mathcal{N} is a direct sum of shifts of summands of $N_{\underline{x}}$ we conclude that the space $\text{Hom}_{\mathcal{N}^{\geq x}}^{\bullet}(M, D_x)$ is a free graded R_I -module for any object $M \in \mathcal{N}$. We define the *diagrammatic character* as follows

$$\begin{aligned} \text{ch} : [\mathcal{N}] &\rightarrow N \\ [M] &\mapsto \sum_{y \in {}^I W} \text{grk} \text{Hom}_{\mathcal{N}^{\geq y}}^{\bullet}(M, D_y) n_y, \end{aligned}$$

where grk denotes graded rank.

Theorem 6.2. *The diagrammatic character gives an isomorphism*

$$\text{ch} : [\mathcal{N}] \xrightarrow{\sim} N$$

as $[\mathcal{H}] = H$ -modules. Under this isomorphism the indecomposable object D_x is mapped to the Kazhdan-Lusztig basis d_x .

Proof. It is clear that ch is a morphism of $\mathbb{Z}[v^{\pm 1}]$ -modules. As explained above, the set $\{D_x \mid x \in {}^I W\}$ gives representatives for the indecomposable objects of \mathcal{N} up to shifts and isomorphism. Hence

$$[\mathcal{N}] = \bigoplus \mathbb{Z}[v^{\pm 1}][D_x].$$

On the other hand, it is immediate from the definition and Lemma 6.1 that

$$\text{ch}([D_x]) = n_x + \sum_{y < x} n'_{y,x} n_y,$$

for some $n'_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. We conclude that ch maps a basis of $[\mathcal{N}]$ to a basis of N , and hence is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$ -modules.

For any expression $\underline{y} = s_1 \dots s_m$, set $d_{\underline{y}} := n_{\text{id}} \cdot b_{s_1} \dots b_{s_m}$. Lemma 6.1 combined with Equation (2.5) implies, by construction of the I -antispherical light leaves, that $\text{ch}([N_{\underline{y}}]) = d_{\underline{y}}$. For any $s \in S$ we have tautologically

$$\text{ch}([N_{\underline{y}}][B_s]) = \text{ch}([N_{\underline{y}'}) = d_{\underline{y}'} = d_{\underline{y}} \cdot b_s,$$

where $\underline{y}' = s_1 \dots s_m s$. We conclude that ch is a map of $[\mathcal{H}] = H$ -modules on the $\mathbb{Z}[v^{\pm 1}]$ -submodule generated by $[N_{\underline{y}}]$, where \underline{y} ranges over all expressions. However this submodule is all of $[\mathcal{N}]$ and hence, ch is an isomorphism of $[\mathcal{H}] = H$ -modules.

We will prove by induction in $l(x)$ that $\text{ch}([D_x]) = d_x$, so let us suppose that we know this equality for all y such that $l(y) < l(x)$. Let \underline{x} be a reduced expression for $x \in {}^I W$. Then in \mathcal{H} we have

$$B_{\underline{x}} = B_x \oplus E,$$

where E is some self-dual object, all of whose indecomposable summands are parametrized by $y < x$. By acting on N_{id} we conclude that

$$N_{\underline{x}} = D_x \oplus \overline{E},$$

where \overline{E} is a self-dual combination of D_y with $y < x$ and $y \in {}^I W$. As observed above, we have $\text{ch}([N_{\underline{x}}]) = d_x$ and so it is self-dual. By induction, $\text{ch}([\overline{E}])$ is self-dual. We deduce that $\text{ch}([D_x])$ is self-dual as well, as the difference of two self-dual elements.

Finally, by the main theorem of [EW14] (more precisely, see second sentence following [EW14, Theorem 3.6]) we know that $\text{Hom}_{\mathcal{N}_{\geq y}}^{\bullet}(D_x, D_y)$ is generated in strictly positive degrees for $y < x$. We conclude that the polynomials $n'_{y,x}$ defined above actually satisfy $n'_{y,x} \in v\mathbb{Z}[v]$ for $y < x$. Hence by the uniqueness of the Kazhdan-Lusztig basis we deduce that

$$\text{ch}([D_x]) = d_x.$$

The theorem follows. \square

Corollary 6.3. *The anti-spherical Kazhdan-Lusztig polynomials $n_{y,x}$ have non-negative coefficients.*

Also, if $J \subset I \subset S$ it is immediate (either from Remark 5.2 or from the fact that \mathcal{N}_I is a quotient of \mathcal{N}_J) that we have a surjection

$$\text{Hom}_{\mathcal{N}_{\overline{J}}^{\bullet}}(D_x^J, D_y^J) \twoheadrightarrow \text{Hom}_{\mathcal{N}_{\overline{I}}^{\bullet}}(D_x^I, D_y^I),$$

and we deduce:

Corollary 6.4. *Brenti's Monotonicity conjecture: $J \subseteq I$ implies that $n_{y,x}^I \leq n_{y,x}^J$, for $x, y \in {}^I W$.*

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