On the role of corporeality, affect and metaphoring in Problem Solving

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Abstract: We explore the role of corporeality, affect and metaphoring in Problem Solving. Our experimental research background includes average and gifted Chilean high school students, juvenile offenders, prospective teachers and mathematicians, tackling problems in a workshop setting. We report on observed dramatic changes in attitude towards mathematics triggered by group working for long enough periods on problem solving and we describe ways in which (possibly unconscious) metaphoring determines how efficiently and creatively you tackle a problem. We argue that systematic and conscious use of metaphoring may significantly improve performance in problem solving. The effect of the facilitator ignoring the solution of the problem being tackled is also discussed.

Introduction

The relevance of problem solving for the teaching and learning of mathematics has become commonplace nowadays. In the Western world this has been triggered to a great extent by the pioneering taxonomy of Polya (1945), as reported in first person by Schoenfeld in Arcavi et al. (1998), Appendix A. Different approaches to problem solving in mathematics and mathematics education have emerged in the course of time (Schoenfeld, 1985, 1992, 2010, 2012; Silver, 1985), some of them having their roots before the 20th century, like the Japanese problem solving approach, described in Isoda & Nakamura (2010), Isoda & Katagiri (2012).

Our main working hypotheses regarding problem solving concern the role of metaphoring, cognitive mode switching and embodiment. More precisely, we claim that metaphoring may arise naturally as a response to a problematic situation the learners are involved in, implying quite often a change in the cognitive mode or style of the learner. Moreover, we claim that corporeality plays also a fundamental role in problem solving since we do not just tackle or solve problems "in our heads" but through body, mind and affect (Hannula, 2012).

Our purpose in this paper is to bring grist to the mill of our hypotheses by presenting various down to earth cases where the implementation of the sort of approach we intend to foster makes a dramatic difference to the learner's understanding, feeling and performance.

To this end we report on some case studies of problem solving with a wide spectrum of learners, ranging from average or gifted regular students majoring in science as well as in social science and humanities to primary school teachers from rural areas in Chile and juvenile offenders engaged in a social re - insertion programme.

Let us recall first some basic facts and references regarding metaphoring, cognitive modes and corporeality.

The “metaphorical approach” we adhere to in mathematics education, has been progressively laid down during the last decades (English, 1997; Lakoff & Núñez, 2000; Presmeg, 1997; Sfard, 1997, 2009; Soto-Andrade, 2006, 2007, 2012, 2013, and many others), as (conceptual) metaphors are not being regarded as simply rhetorical devices as they classically were, but as powerful cognitive tools helping us to build or grasp new concepts, as well as to solve problems in an efficient way.

Well known examples of conceptual metaphors in mathematics education are: “subtraction is going backwards”, “an equation with one unknown is a balanced pair of scales with one incognito weight”, “probabilities are weights, or masses”, “a random walk is a fission process, or an iterated splitting or sharing”, “a polygon is a closed space between crossing sticks".

The concept of cognitive modes, or "cognitive styles" in French, emerged from work by Luria (1973) and was further developed by Flessas (1997) and Flessas & Lussier (2005), who pointed out to their impact on the teaching-learning process. A cognitive mode is defined nowadays as one's preferred way to think, perceive and recall, in short, to cognize. It reveals itself particularly in problem solving. To generate what they call the 4 basic cognitive modes, Flessas and Lussier (2005) combine 2 dichotomies: verbal – non verbal and sequential – non sequential (or simultaneous), closely related to the left – right hemisphere and frontal – parietal dichotomies in the brain (Luria 1973). This affords 4 basic cognitive modes: *verbal-sequential, verbal-simultaneous, non-verbal - sequential, non-verbal-simultaneous.* This may be supplemented with Schwank's dichotomy *predicative - functional,* also described as *structural-dynamic* (Schwank, 1999) to provide 8 cognitive modes in all.

As said before, one of our hypotheses is that the most meaningful and significant metaphors arising in a problematic situation will involve a cognitive mode switch for the learner. Moreover, we hypothesize that the ability to switch from one way of cognizing to another is trainable.

Regarding corporeality, one of our basic tenets, i. e. the importance of bodily attitude (e.g. standing and working on non permanent vertical surfaces) in cooperative problem solving has alreadynbeen highlighted by Liljedahl (in press). For an authoritative survey on the role of affect in problem solving, we refer to Hannula (2013).

We now proceed to present our case studies.

A. Problem solving by juvenile offenders: a multiple case study

A big challenge in Chilean society is the re-education and re-insertion of juvenile offenders, guilty of various felonies as well as misdemeanors. This challenge is being addressed, among other actions, by a joint program run by the National Office for Minors (SENAME: Servicio Nacional de Menores) and the Faculty of Science of the University of Chile, that involves mathematical training workshops held at the University for small groups of minors from SENAME. Usually these minors are dropouts from high school whose education is scanty and fragmentary, to say the least.

To work with these young persons (aged 18 to 22) we have implemented a highly metaphorical, enactive and visual approach (Presmeg 1997, 2006; Soto-Andrade, 2006, 2007, 2012, 2013), eliciting a much higher motivation than traditional teaching.

We report here on the outcomes and performance of the juvenile offenders in a workshop carried out in 2011-2012 and 2014, where an open-ended Finnish problem (Pehkonen, 1995) was proposed. The workshops. lasted one semester and they had 7 to 10 students.

Our main specific working hypothesis was that student centred, open-ended and creative problem solving activities, that can relate to the personal and social needs of the pupils and their past experiences, in the sense of Pehkonnen (1998) and Järvinen & Twyford (2000) were especially suited to the case of young subjects, like our juvenile offenders. Since they have developed remarkable skills to survive in hostile or repressive environments, we hypothesized that creativity and metaphorical activity may be more spontaneous in them than in “regular” students, and that emerging idiosyncratic metaphors might be of significant help for them to solve the challenges proposed.

The methodology consisted in observing and interviewing the students as they carried out the activity described below. Records of this observation comprised videos, written and drawn production of the students and some transcriptions.

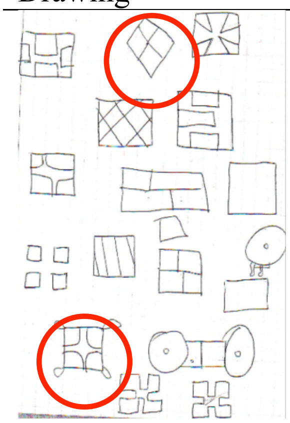
In 2011,2012 and 2014, We carried out a 60 minute work session on the following Finnish open-ended problem (Pehkonen, 1995):

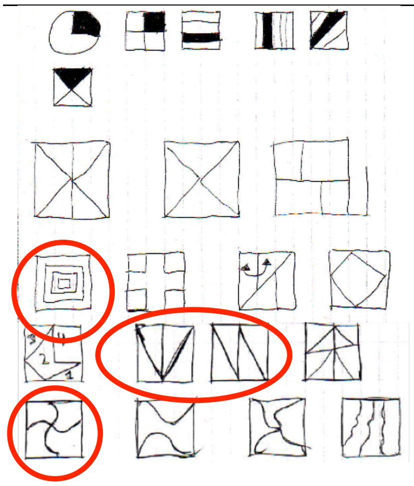
Partition a square in four equal (i.e. congruent) pieces in four different ways.

A sample of solutions figured out by the juvenile offenders in 2011-2012 is given in Figures 1 and 2 below. A whiteboard of solutions obtained in 2014 is shown in Figure 3 below.

First, they found quickly the most obvious 3 ways to partition the square and realized that horizontal and vertical stripes were “the same”, but they had a hard time finding a fourth (essentially) different way. During a lapse of approximately 20 minutes, they generated however an interesting array of wrong answers (see Figures below), each on his own. Since no correct solution emerged for a while, the facilitator of the workshop (JSA) had the idea to share their wrong solutions on the whiteboard, in particular the “absurd” concentric squares solution shown below.

Figure 1: Partitioning the square, juvenile offenders 2011-2012.

Drawing	Comments
	Juvenile offenders: Claudio 1 Notice the incomplete partition.



Claudio 2

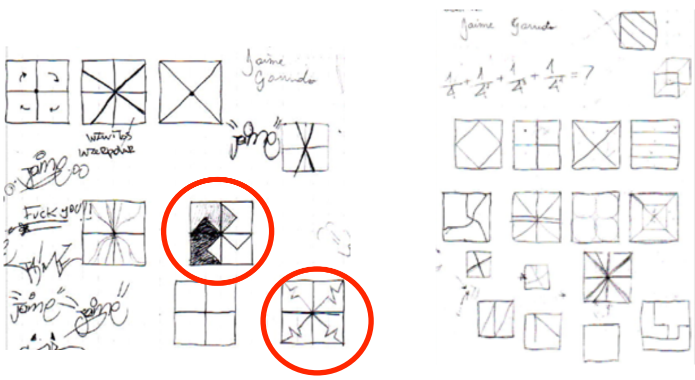
Notice the absurd concentric solution, with central symmetry

Two classical solutions.

A remarkable solution with central symmetry

Shortly afterwards, one student got the idea of drawing a line upwards from the centre of the square to the border, “deviating from the straight line upwards, turning a bit to the left” in his own words. Notice here the heavy metaphorical content of this description, that applies to his own condition: in Spanish, indeed, “desviarse del camino recto” (“to deviate from the straight - or righteous - path”) is a very common expression. Then all sorts of “deviations” popped up (see Fig. 1 above and Fig. 2 below), providing a handful of different solutions. One offender, who had worked this out independently, when asked how did he get the idea of going out from the centre of the square, answered: “I got it from the wrong concentric squares solution, but for me the centre is not an origin, but a ‘punto de fuga’ ” (a usual expression in perspective drawing, in Spanish, meaning literally “escape point”). He explained then that the wrong solution of his mate appeared as a framed aisle in perspective to him, and that he very much liked to draw in perspective. Notice again the metaphor for his life condition…

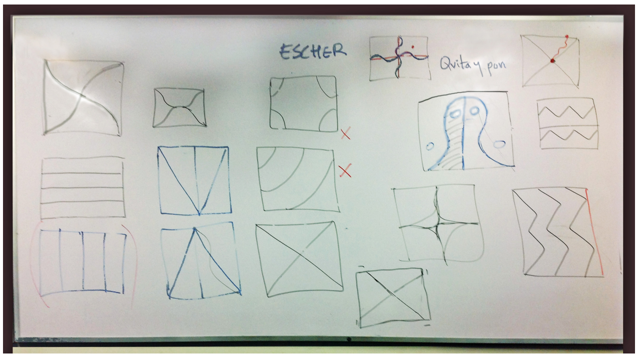
Figure 2: Jaime’s partitions.

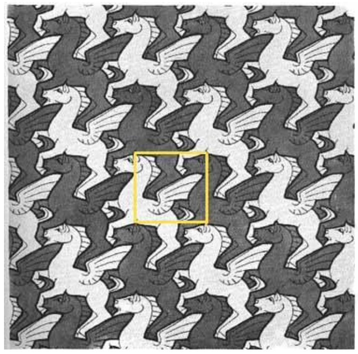


Others like Jaime (Fig. 2 above) also realized that infinitely many solutions may be obtained by “adding and taking away” (first red circle in Fig. 2), instead of deforming the straight paths from the centre to the border. Notice however his very original partition in the second red circle, obtained by “perturbing the straight line with a shiver, or a frisson”. In this case, we also see some evidence of his affective mood and tensions in his writings next to the squares. This suggests a close relationship between creativity and affect and emotion (Hannula, 2013)

One interesting phenomenon, is that among these juvenile offenders, dropouts from school, who perform very poorly in standard (TIMSS like) assessment tests, the same clever idea emerged (take any path from the centre of the square to its border, and rotate it thrice in a quarter of a turn!) as the one Ragnar (a case study in Soto-Andrade, 2006, who majored later in anthropology) had during a work session on this problem with Prof. Pehkonen himself, 2 years ago, in Santiago. The point is that Ragnar's cognitive and educational background (starting at a Waldorf school) is wide apart from our juvenile offenders'.

Figure 3. Square Partitions, Juvenile Offenders 2014. Figure 4. Escher's Pegasi





Notice the incorrect curved partition in the lower right corner. It has the merit however of being the first partition suggested in this session that used curved lines! Its author was reluctant to share it, because another offender pointed out immediately that it was wrong, but he realized afterwards that this partition opened up the way to many correct curved partitions! In particular the one provided by the hooded figure – "encapuchado" in Spanish - in the second row. It was christened that way by his author, as a humorous allusion to hooded youngsters that after a pacific civil demonstration often initiate riots by throwing stones and burning devices to police forces (something familiar for these young offenders).

Moreover they also remarked, working in an interactive way, that new partitions may be obtained from very simple ones, like the one of four squares, by "stealing away" a bit from one square on one side and "giving it back" on the other side. So they rediscovered by themselves Escher's method of tessellation by compensation, and were able to figure out very quickly how to construct the tessellation shown in Fig. 4, where the superimposed yellow square tile was figured out by them and not given in advance!

B. Problem solving by university students: Multiple case studies.

When working with these second and third year university students we tried hard to choose exciting and hard problems and to reinforce sense of humor in the classroom. This motivated the students significantly.

We report here on some important aspects of three workshops carried out in 2013 and 2014. Our main specific working hypothesis was that group work is extremely important when trying to solve difficult problems (much more than in other contexts) mostly because what is crucial in solving hard problems is a meta-mathematical attitude related partly to self-esteem and partly to "know what to do when you don't know what to do".

The background for our experimental research was the following. Each one of the three experiences was a one semester workshop (3 hours a week) for second or third year university students majoring in mathematics or mathematics education. The first workshop had 30 students, the second one 7 and the third one 10.

The methodology consisted in observing and interviewing the students as they carried out the activity described below. Records of this observation comprise, written and drawn production of the students and some written observations of them.

The work methodology that we are going to explain was developed over the years in different workshops of problem solving and problem invention. In particular, one important workshop that will not be described here took place in 2001 in a high school. Just one year after the workshop the three participating students obtained the three gold medals in the Chilean mathematics olympiad. That was an important moment, where the facilitator realized that this methodology had some interest.

Our working methodology was: work sessions were 180 minute long. Students worked in self defined groups, standing up, in front of a blackboard, the facilitator behind them. They worked on hard problems, that needed a whole 180 minute work session to be solved. More or less half of the time, problems remained unsolved at the end of the work session. Answers or solutions were never *given* by the session facilitator (NL). Hints towards a solution were only given when the students looked demotivated. Problems were selected because of their amusing and interesting character (to foster motivation among the students) according to the facilitator's appreciation, and then, in the following workshops, the opinions of the students of the first workshops were taken into account. Half of the time the facilitator did not know the solutions of the proposed problems. When a group quickly solved the problem, they were asked to generalize the problem or to find a variation.

Examples of problems given in the first sessions are:

1. Resolve and generalize the "towers of Hanoi" problem.
2. The SEND+MORE=MONEY problem.
3. The prisoners and hat puzzle in the ten-hat variant, and generalizations.

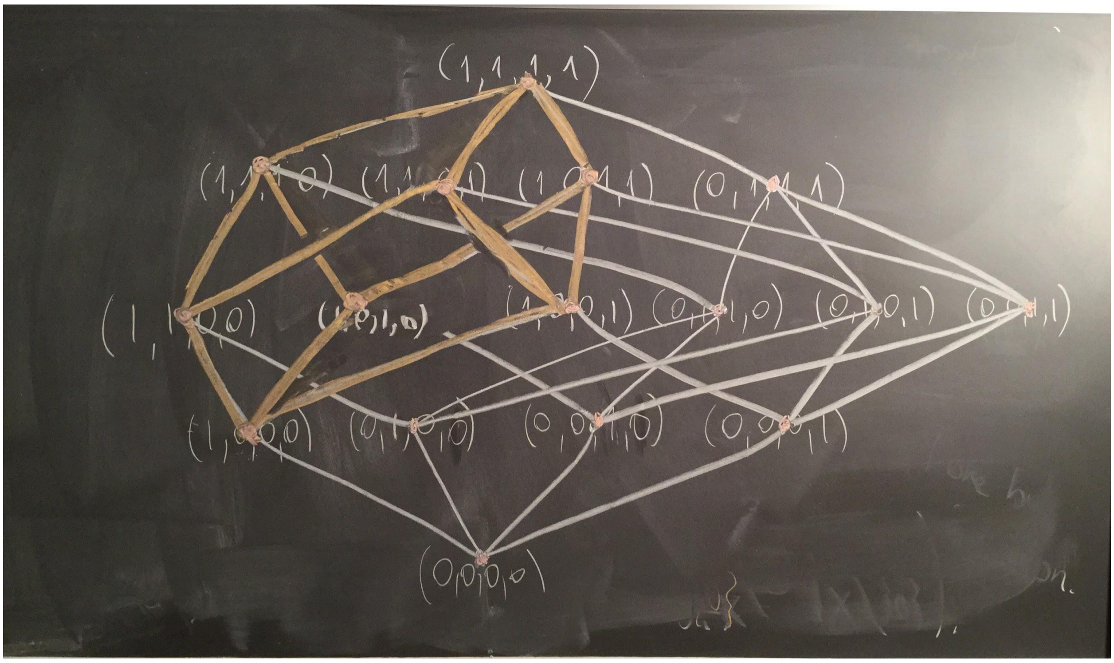
Motivation is one of the main driving forces for development in mathematics and can be developed in different ways, one of which is to find beauty in mathematics. But some students have to enact situations in order to "see" the beauty. We give here one example.

The facilitator gave the following problem: prove that the sum over "faces" of all dimensions of an hypercube is 3 to the power of the dimension of the hypercube. For example in a usual cube, the sum of the vertices (8) plus the number of edges (12) plus the number of faces (6) plus the number of cubes (1) is 27, that is, 3 to the third power.

The idea is that this is a good lighthouse to discover what a hypercube of more than three dimensions is. Students had many different approaches to this problem but one group (we call it group A) was confused about the idea of what a “face” in a 4-hypercube looked like. They imagined that it should look like a usual cube but they were not able to “see” this. At some point, one of the students in group A said “in a cube, a face is *all the ways to walk* from one vertex to another vertex that is two steps apart”. So they used the metaphor that the cube is a “place” where you can walk, and “objects” (for example, faces) inside this place are defined by your possible movements. This follows exactly the two patterns that Thurston teaches us (Thurston 1998) when he explains how to imagine 3-manifolds: you have to imagine yourself inside of the manifold and imagine that you are more or less the same size of the geometric object you try to imagine.

Then they drew a 4-hypercube in the blackboard by using a natural definition of the 4-hypercube, sequences of four 0’s and 1’s, where an edge joins two vertices if the corresponding sequences differ in exactly one position. Finally they drew in orange all the “ways to walk from one vertex to another that is three steps apart”, as one student said, and suddenly they “saw” a 3-cube (a face) inside the 4-hypercube as shown in Fig. 5.

Fig. 5.



Something remarkable about that moment was that one student remained completely silent with eyes wide open. She then told the facilitator that it was the first time she had ever experienced something beautiful in mathematics. After that moment her attitude towards mathematics changed dramatically and she became, until the present day, much more interested in mathematics in general.

A second remark about these workshops was internal to the facilitator (NL) and has to do with the implicit set of metaphors underlying positioning in the classroom. The facilitator had set up a “classical classroom” where students were sitting on their chairs and the professor wrote on the blackboard. The implicit metaphor in that situation is that the professor has “something to give to the students” while in the setting explained before a reasonable metaphor of what is happening is “the professor is behind the students to support them if they fall”.

The important point about this is that the facilitator felt, in the first case (classical classroom) that students were not smart in general, because they had a hard time understanding the theorems. While in the second case, the facilitator was mostly impressed by how smart the students were, because their creativity came into play and they invented lots of solutions of a different nature to those that the facilitator would have think of. This makes a huge difference in the motivation the students will develop, because the opinion the teacher has about their students is something intangible but somehow understood by them.

Another interesting thing to be remarked was the change in attitude towards the problems. More or less in the fourth session of every workshop the students stopped being demotivated if they didn't solve the problem after a few minutes; they began to realize that the objective of the work sessions was to think all together, and not to immediately solve the problem. Many times they stayed more than the three hours just because they were curious. One girl once told the facilitator that even though she liked the workshop, she was thinking to stop coming because Monday night (the workshop was held on Mondays) she could not sleep if she had not solved the current problem.

Reflecting on the role of corporeality in these workshops: the fact that students were standing up for three hours (with the possibility of sitting down from time to time) was quite important for them. Many of them bear witness that this was in part the reason why they were so active in the sessions. Some groups that sat down reported to have much less and slower communication between them. Also we remarked that they tend to have a contemplative attitude towards the problem if the problem is hard and to lose concentration (having other students thinking with you seems to increase concentration levels).

Another instance of corporeality involved problems related to surfaces. In one problem, the students were asked to visualize the geometry of the “Whitney umbrella” (the zeros of the equation $x^2=y^2z$) . For a large number of students the task of imagining the surface is impossible without gesturing. For example, one group explained the surface to one another by moving their hands outwards, with joined hands corresponding to the locus z = constant. It is also interesting whether the students "see" this surface as an umbrella (again a metaphor).

One activity in one of the workshops was to play a game where metaphors, corporeality and affect were deeply intertwined. It is the following game (invented by Sebastian Libedinsky) that we call SL game. There are two players. They both have 30 “objects”. Each round they have to say a natural number (including zero) bounded by 30, both at the same time. The biggest number wins the round. Now for the next round each player "loses" the number of objects he said in the first round. For example if player one says 7 and player two says 4 in the first round, then player one wins the round but he has only 23 objects left for the next rounds, while player two lost the round but he/she has 26 objects for the next rounds. In each round you cannot say more than the number of objects you still have. The player that first wins three rounds, wins the game.

This game was very fun to play for the students and they laughed a lot playing it, so that their sense of humor was highly activated. The atmosphere allowed interesting phenomena to arise. For example each player's strategy depended on what metaphor he used for the objects (matches, water, fingers, etc). They had to choose one in order to play, but they usually chose one because they thought it was funny. Most of them did not realize that their choice would be so determinant. For example, players who imagined objects as being water were more likely to invent probabilistic strategies. Players imagining the object as matches were much more likely to try to win the first round. One student said laughing: "I want to start a fire". The players using their fingers to represent objects were more likely to say little numbers (for evident reasons). The game was usually won by players applying some probability rule, so "water" usually won over "fire". Another important feature of this game is that less gifted students usually beat the more gifted students.

Let us make a final remark regarding these workshops. Many times the facilitator did not know the answer of a proposed question. Not knowing the answers entailed a lot of fun for the facilitator, which translates in a good state of mind of the whole class. He didn't try to solve the problems with the students but he helped them in developing their ideas without knowing himself if a particular way they engaged in would lead to a solution. We believe that the fact of ignoring the solution of the problems was extremely important for the dramatic change in attitude the students had, because they saw a lot of meta-mathematical reasoning from the facilitator. He really pondered with them whether their approaches had chances to succeed. If he had known the answer beforehand, they could have never seen a honest reasoning of this type from him.

Finally, in these workshops we asked for problem posing. We can remark that when students try to invent problems they quickly appreciate how difficult it is to find a problem which is both feasible and pleasant for them and their fellow students. This leads them to appreciate much more the problems posed to them by the teacher. They start developing sensitivity towards the art of creating or modifying problems. And this is a key to deep mathematical thinking, as can be seen by the fact that in mathematical research half of the problem is to find a fascinating problem, that usually comes in the form of conjecturing a fascinating phenomenon. But to obtain this you have to solve other problems beforehand.

As an example of this point, there was a group of students that realized that in the power set of a finite set, the operations "intersection" and "symmetric difference" are like adding and subtracting in "certain ring". This was a starting point for many natural questions about ring theory that they tried to solve in the course of the workshop.

C. Problem solving by primary school teachers: What about the sum of all exterior angles of a polygon?

We worked on this problem in 2010 in a session of a professional development program in Puerto Montt, Chile, with three groups of 30 in-service primary school teachers coming from rural areas in the South of Chile.

The usual approach to this problem, observed among secondary school teachers in our country and elsewhere, is to calculate first the sum of interior angles of the polygon, that depends on its number of sides, then express exterior angles in terms of interior angles, calculate diligently and finally arrive to the conclusion that the requested sum measures 360° and say, together with their students: We are done! We however share Schoenfeld's claim that we are not done! (Schoenfeld, 2012). To explore other ways to approach the problem, we suggested to the teachers to metaphorize the ingredients of the problem. That is, to try to figure out different metaphors for a polygon, to begin with. Then, do the same for the exterior angles, which they found almost unanimously less friendly than the interior angles.   After 10 to 15 minutes, some interesting metaphors emerged, like the following:

- A polygon is an enclosure bounded by crossing sticks ! (Fig. 6)
- A polygon is a "German closed path", i. e. a chain of rectilinear segments, no curves…

These metaphors were often expressed in gestural language.

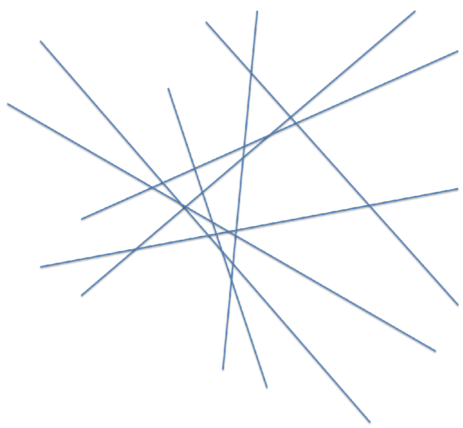


Figure 6. Crossing sticks metaphor for a polygon

The teachers enacted then these metaphors. When enacting the first one with 20 cm long sticks, they got the idea of moving the sticks, sliding them first, to better see the exterior angles! (Fig. 7), then moving them parallel to themselves, so as to reduce the size of the selected enclosure, preserving its shape! An idea more likely to emerge when you enact your metaphor with concrete material, than when you work with pencil and paper or just recite the scholastic definition of a polygon. In this way they "saw" immediately that the sum of exterior angles is a whole angle.

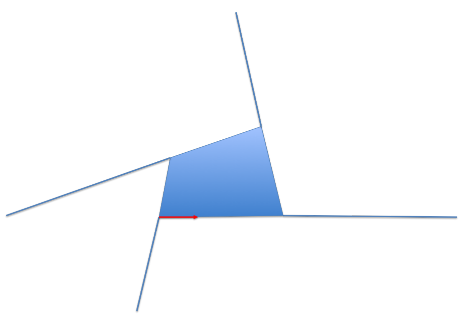


Figure 7. Sliding the sticks…

They also enacted the second metaphor: one of them gave instructions to another one to follow a given polygonal closed path: begin here, go straight ahead 5 steps, stop, then turn to your left in 45° (an exterior angle, much more relevant in this context than the interior angle!), then go on for 8 steps, etc. until the walker came back to its initial position with her nose pointing in the same direction as when she started. This teacher "felt" then, that she had made in all a whole turn! We see then a friendly, metaphoric and enactive way to figure out the sum of all exterior angles of (any) polygon.

D. Discussion

We have seen that these approaches elicit positive cognitive and affective reaction from our students (juvenile offenders, university students and primary school teachers) . After the workshop sessions, they bear witness to a completely new experience of mathematics, when comparing with their previous mathematical instruction.

In the case of juvenile offenders, they appear as a group far more creative and autonomous than regular students and also teachers, with the exception of students like Ragnar, who have had a first rate educational and cognitive experience since his childhood. This convergence of bright ideas, emerging independently in subjects so wide apart in personal life histories, socio-economical status and educational studies, when confronted with open ended problems, with a strong visual, motoric and metaphorical component, is a phenomenon that deserves further study, in our opinion. It is also remarkable, how metaphors emerging from their life condition (breaking the law, punishment, full time and part time imprisonment, etc.), like "deviating from the straight path" or taking advantage of an "escape point" play the role of tools helping them to solve the proposed cognitive challenge. Our findings also suggest that further research should be carried out on the cognitive and therapeutic effects of the metaphorical approach to mathematical challenges in juvenile offenders engaged in a re – insertion process as well as on exploring various means to free the expression of creativity in regular students.

We argue that the examples shown in this paper (hypercube, exterior angles of a polygon, Whitney umbrella, objects in SL game) show how crucial to problem solving the role of enactive bodily metaphoring might be.

Notice that also in the problem of partitioning the square, when you try to enact the procedure of partition, you may realize that you are unconsciously metaphoring it as slicing a pizza with a knife, i.e. doing straight cuts, not curved ones. An alternative metaphor is however dividing a paperboard square with scissors, that opens up the possibility of curved portions, and so on...

The other important part of corporeality is how important it seems to be in long problems. We have remarked that standing up makes the whole body work and this bodily attitude helps concentrating on the problem and also helps the students to be more courageous regarding the ideas they have. It is also easier to share their excitement about a new idea they had, since they can move more easily and excitement is usually shared through body language.

We think that to work in groups while solving a problem has usually various advantages. A fundamental part of problem-solving is self-esteem, and it is a long and hard individual process to develop it. If you work on hard problems in groups, your fellow students help in this sense, because they are excited every time you have a good idea, probably more than the teacher would be, even if he is very sensitive, because as sensitive as he can be, he will never be able to understand in detail what things are difficult for her students.

The fact that the teacher tries to go with the flow of student's thoughts (not imposing his own way of solving a problem) usually makes him discover, as we said before, how creative students can be. The resulting teacher's excitement, honestly communicated to the students, turns out to be very inspiring for them, because excitement in this kind of matters is difficult to fake. It is very important that the teacher feels and express admiration of his students (a point strongly emphasized in Japanese problem solving approach). It is very important (and usually underestimated) in a lesson that both the teacher and the students are motivated. The fact of not knowing the answer to the problems proposed makes the job of the teacher especially stimulating and fun, but we believe that for the teacher to feel comfortable doing this, he has to have a good level in solving problem, ideally much higher than the students.

One thing we have observed is that the difficulty level of a problem is a fundamental issue in problem-solving. Someone can be very good in solving "easy problems" (problems that take him 20 minutes or less to solve) but very bad in solving harder problems or even extremely easy problems (3 minute problems). We have seen examples of people being extremely good at solving hard Olympiad problems (that can take them a couple of hours) and quite poor to solve research-level problems or very easy problems. We believe that in a class it is paramount to have all levels of problems, because they use different skills for different levels and they can be motivated in different ways. We hypothesize that the fundamental differences in responses to these different classes of problems is due to self-esteem and the frustration threshold.

We claim then that by working on problem solving with enactive metaphors like these, learners of all walks of life may generate their own creative approaches and think mathematically, something that otherwise would be accessible just to a happy few…

We would like to finish by saying that the expression "problem-solving" seems to be a bad metaphor for what students might do. It implies that the problems are "outside" in the world and that their relationship to them has to be to solve them. Something like a gatherer, eating the food he finds in the wilderness. We prefer the expression "problem development" or "problem looping" that suggests the idea that problems are constantly solved and invented in a circular never ending process involving subjects and a world that co-determine each other (Varela, 1987, 1999)

References

Arcavi, A., Kessel, C., Meira, L., and Smith, J. (1998). Teaching mathematical problem solving: A microanalysis of an emergent classroom community. In Alan Schoenfeld, Ed Dubinsky, and James Kaput (Eds.), *Research in Collegiate Mathematics Education III* (pp. 1–70). Providence, RI: American Mathematical Society.

English, L. (Ed.) (1997). Mathematical reasoning: Analogies, metaphors, and images. Mahwah, NJ: Lawrence Erlbaum.

Flessas, J.: 1997, L'impact du style cognitive sur les apprentissages, Revue Éducation Francoph., 25, n° 2, http://www.acelf.ca/c/revue/revuehtml/25-2/r252-03.html

Flessas, J., & Lussier F. (2005). "La neuropsychologie de l'enfant". Paris: Dunod.

vom Hofe, R. (1995). Grundvorstellungen mathematischer Inhalte. Heidelberg: Spektrum Akademischer Verlag.

Hannula, M. (2013). Affect in Mathematics Education, in Encyclopedia of Mathematics Education, S. Lerman (Ed.), pp. 23-27, DOI 10.1007/SpringerReference_331465

Isoda, M. & Nakamura, T. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education vol.92, No.11, 1-157

Isoda, M. & Katagiri, S. (2012) Mathematical Thinking: How to develop it in the classroom, Singapore: World Scientific.

Järvinen E.-M. & Twyford, J.(2000): The Influences of Socio-cultural Interaction upon Children's Thinking and Actions in Prescribed and Open-Ended Problem Solving Situations. International Journal of Technology and Design Education. Dordrecht: Kluwer Academic Publishers. 10, 21–41.

Lakoff, G., & Nuñez, R. (2000). Where Mathematics comes from. New York: Basic Books

Liljedahl, P. (in press). The affordances of using visually random groups in a mathematics classroom. In Y. Li, E. Silver, & S. Li (eds.) Transforming Mathematics Instruction: Multiple Approaches and Practices. New York, NY: Springer.

Luria, A.: 1973, The working brain, N. Y. : Penguin.

Pehkonen, E. (1995): Introduction: Use of Open-Ended Problems. ZDM 27 (2), 55–57.

Pólya, G. (1945; 2nd edition, 1957). How To Solve It. Princeton: Princeton University Press.

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), Mathematical reasoning: Analogies, metaphors, and images (pp. 267-279). Mahwah, NJ: Lawrence Erlbaum.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez, & P. Boero (Eds.), Handbook of Research on the Psychology of

Mathematics Education: Past, present and future, (pp. 205-235). Rotterdam: Sense Publishers.

Schoenfeld, A. H. (1985) Mathematical Problem Solving. Orlando, Fl: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Douglas Grouws (Ed.), *The handbook of research on the teaching and learning of mathematics* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (2010). How We Think: A Theory Of Goal-Oriented Decision Making And Its Educational Applications. New York: Routledge.

Schoenfeld, A. (2012). Social dynamics for supporting creative mathematical thinking and problem solving, International Congress in Mathematical Education, Seoul, Korea, TSG22-6, 4478-4487.

Schwank, I. (1999). On Predicative versus Functional Cognitive Structures, European Researcy. In Proceedings of CERME 1 (The First Conference of The European Society for Research in Mathematics Education), pp. 84-96.

Sfard A. (1997). Commentary: On metaphorical roots of conceptual growth.In L. English (Ed.), Mathematical reasoning: Analogies, metaphors, and images (pp. 339-371), London: Erlbaum.

Sfard A (2009) Metaphors in education. In: Daniels H, Lauder H, Porter J (eds) Educational theories, cultures and learning : a critical perspective. Routledge, New York, pp. 39 – 50.

Silver, Edward A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In Edward A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 247-266). Hillsdale, NJ: Erlbaum.

Soto-Andrade, J. (2006). Un monde dans un grain de sable: Métaphores et analogies dans l'apprentissage des maths. Ann. Didactique Sciences Cogn. 11, 123– 147.

Soto-Andrade, J. (2007). Metaphors and cognitive styles in the teaching-learning of mathematics. In D. Pitta-Pantazi, & J. Philippou (Eds.). Proceedings CERME 5 (pp. 191- 200). Retrieved May 21, 2008, from http://ermeweb.free.fr/CERME5b/

Soto-Andrade, J. & Reyes-Santander, P. (2012) Mathematical Cognition in Juvenile Offenders : a Case Study, en Proceedings *ICME 12* (July 2012, Seoul, Korea), pp. 4721-4730.

Soto-Andrade J. (2013). Metaphors in Mathematics Education. In: Lerman S. (Ed.) Encyclopedia of Mathematics Education: SpringerReference (www.springerreference.com). Springer-Verlag Berlin Heidelberg, 0. DOI: 10.1007/SpringerReference_313292 Retrieved: 2013-10-01

Thurston W. (1998). How to see 3-manifolds. Classical and Quantum Gravity (**15)** 2545-2555

Varela, F.J. (1987). Lying down a path in walking. In W.I. Thompson (Ed.). Gaia: A Way of Knowing, Hudson, NY: Lindisfarne Press, pp. 48-64.

Varela, F. J. (1999). Ethical know-how: action, wisdom, and cognition, Stanford: Stanford University Press.